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CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

EDITED BY

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an annular surface is obtained having plane lines of curvature whose order is half that of the surface—the latter not being of revolution. The scrolls are also of particular form; their asymptotic lines are of order only half that of the general scroll of the same type.

Professor Morley's second paper is in abstract as follows: It is known that the bitangent planes of a rational space quartic R_4 touch a quadric, which contains the tangents of the curve at the flexes or points of contact of the four stationary planes. Any tangent plane of this quadric cuts R_4 at four points U, such that the Hessian points H of U also lie in a plane. This plane again touches the quadric. Any tetrad $U + \lambda H$ of R_4 lies in a plane which touches the quadric. All the planes form a pencil whose axis is a generator of the quadric, of the system to which the flex lines belong. Among the planes of the pencil there are two which cut R_4 in a self apolar tetrad, three which cut R_4 in apolar pairs of points and three bitangent planes. The one set of three planes is the cubicovariant of the other; the two planes form the common Hessian of the two sets of three planes.

Professor Gordan's paper, which will be published in the *Transactions*, is in brief an exhaustive discussion of the curves of order or class 3, which have for their Hessian or Cayleyan a given curve of the same order or class. The author shows that these number 3 in general, but in special cases 1 or ∞; for each case the types are determined explicitly.

The point of departure for Dr. Hawkes's paper was furnished by a consideration of certain criticisms of Benjamin Peirce's Memoir on linear associative algebra, by Study, Molien, Cayley, and others. The paper shows how Peirce's methods can be naturally extended so as to arrive very directly at an enumeration of all number systems in less than six units, which are inequivalent, non-reciprocal, irreducible, and possess moduli.

Of the many theorems contained in Sturm's famous memoir of 1836, the two Theorems of Comparison may be regarded as most fundamental. Professor Bôcher has shown on other occasions (cf. BULLETIN, April 1898 and Dec. 1899, p. 100) how the methods which Sturm used for establishing these theorems can be thrown into rigorous form. In the present paper a simpler and more direct method of establishing these theorems is given along lines suggested by passages, to which Professor H. Burkhardt kindly called the author's attention, in one of d'Alembert's papers on the vibration of strings (*Memoirs of the Berlin Academy*, volume 70 (1763), p. 242). The fundamental idea involved is the reduction of the linear differential equations of the second order to Riccati's equations by the well known change of dependent variable.

The problem discussed by Professor Smith is a generalization of the question treated by Moutard in 1864, viz., to find properties of surfaces enveloped by spheres intersecting a fixed sphere at right angles. The generalization is this: The surfaces to be studied are enveloped by spheres intersecting a fixed sphere at any constant angle. As distinguished from the problem discussed by Moutard, the question is one of the higher spherical geometry, in that the algebraic sign of the radius of a sphere must be regarded. This amounts to distinguishing between the two sides of a surface, as given, *e. g.*, by the two directions of the normal; and this distinction is indicated by the term, "oriented surface." The treatment of the problem is purely geometrical, being based upon a certain simple contact transformation, here defined for the first time. Adopting Sophus Lie's terminology, "linear spherical complex," for the aggregate of ∞^3 oriented spheres intersecting a fixed fundamental oriented sphere under constant angle, this definition is as follows: Given an oriented sphere S, then all ∞^2 oriented spheres tangent to S and belonging to a spherical complex A are tangent to a second oriented sphere S', and S is said to transform into S' by inversion in the complex A. The transformation is involutory. This general inversion depends upon five essential parameters, and includes as particular cases inversion in a sphere, the parallel transformation, symmetry in a plane, and the transformation by reciprocal directions originally defined by Laguerre in 1881. The ∞^3 points of space constitute a particular spherical complex Ω, and inversion in Ω merely turns over an oriented plane.

The relation of this general inversion to the general sphere-sphere contact transformation of fifteen essential parameters is determined by the theorem: The latter transformation is compounded either of a general inversion, a spherical inversion and a displacement, or these three compounded with inversion in Ω. Various theorems on the general inversion are established, *e. g.* (1) A general inversion A is uniquely compounded of a parallel transformation

D, a spherical inversion I and the same parallel transformation D, *i. e.*, $A = DID$. (2) A general inversion A is compounded in two ways of a general inversion A_0, the inversion in Ω, and the same inversion A_0, or $A = A_0 \Omega A_0$. The last theorem leads to a representation of the ∞^3 oriented spheres of the complex A by the ∞^3 points of space, in that the former transform into the latter by inversion in A_0. This representation is fundamental for the point of view adopted in the paper. The following result is now established:

A surface F enveloped by ∞^2 (∞^1) spheres belonging to the spherical complex A is transformed by A_0 into a non-oriented primitive surface (curve) $f(c)$, regarded as the locus of its points. The order and class of F are equal, and its entire section with the plane at infinity is along the imaginary circle. The centers of these spheres of A lie on a surface F_0 (curve C_0), which is the transformed of $f(c)$ by a birational transformation whose fundamental curve is the imaginary circle. The characteristics of F, F_0, and C_0 are easily determined from those of f and c. The intersection of the fundamental sphere of A and the primitive is a double line (point) on F, while the remaining intersection of F and the fundamental sphere is a line of curvature on F.

Particular interest attaches to the case where the primitive is a general cyclide without nodes. Then F is the so-called surface of singularities of the general quadratic spherical complex, and the following results are established:

The surface of singularities of the general quadratic spherical complex is of order and class 24, and contains the imaginary circle at infinity twelve times. It is sibi-reciprocal under six commutative general inversions A_i, with each of which is associated a cyclide f_i, from which as primitive the surface may be derived. The surface contains six double lines, sphero-quartics, the intersections of the f_i and the fundamental spheres, and is the envelope in six ways of ∞^2 spheres belonging to an A_i whose centers lie upon a quartic surface F_i containing a double conic. The six F_i are confocal, being inscribed in the minimum developable of order 24 circumscribing F, which is also the singular minimum developable of that surface. Thus the focal lines of the F_i are the focal lines of F. The surface F is enveloped by six systems of bitangent planes each belonging to a developable of order 8 and circumscribed to a sphere concentric with the corresponding fundamental sphere along a sphero-quartic. The lines of curvature of F are algebraic, being the transformed of the lines of curvature of the primitive cyclides f_1 by the six inversions which transform the f_1 into F.

When the primitive is a circle (or straight line), the surface F is the cyclide of Dupin. This leads to a very simple derivation of the properties of this surface. Some special cases departing from the general results are discussed. The problem is taken up also for the plane, the case in which the primitive is a bicircular quartic giving results analogous to that just given for the cyclide, the order and class being 16 and each circular point octuple.

It is pointed out that in papers on subjects related to that under discussion the method adopted has been that of deriving results from known theorems in line geometry by the line-sphere transformation of Sophus Lie. To this there are two objections: (1) Spherical geometry is essentially metric; and those theorems in line geometry which lead by this correspondence to such results as are developed above are not *a priori* evident. (2) The correspondence fails except for the case of three dimensions. The discussion of this paper, operating entirely in the realm of spherical geometry, gives results which may be immediately generalized for n dimensions. Thus, *e. g.*, the relation of the problem treated in the paper to the minimum projection of points in a space of $n + 1$ dimensions into the oriented spheres of space of n dimensions is determined.

Goursat's memoirs [*Acta Mathematica*, volume 4 (1884); *Transactions*, volume 1 (1900)] in proof of Cauchy's integral theorem

$$\int_C f(z)\, dz = 0$$

without assumption as to the continuity of the derivative $f'(z)$ on the closed region bounded by the curve C of integration lay deeper foundations for the Cauchy-Riemann theory of functions of a complex variable. One finds an abstract of these memoirs in the BULLETIN for June, 1899, pp. 427–429. Goursat sets out by a direct process to evaluate the integral in question. In Professor Moore's paper, which will appear in the *Transactions*, the integral is by an indirect process proved to have the value zero. The essential elements of the proof are those of Goursat's first paper; by the modification indicated one avoids the necessity of using the lemma to the proof of which Goursat's second paper is devoted.

The following is a summary of Professor Osgood's paper, which appeared in full in the July number of the *Transactions:* The problem of mapping the interior of a simply connected plane region T conformably on the interior of a circle depends for its solution on the proof of the existence of the Green's function corresponding to T. Hitherto the existence of this function has been established for regions T bounded by a finite number of pieces of analytic curves, and, more generally, for regions bounded by a finite number of pieces of regular curves, *i. e.*, curves whose tangent turns continuously as a moving point p traces out the curve. That the boundary of a general simply connected region T may be of an essentially more complicated character is shown by the following example: On the axis of abscissas assume at pleasure a set of points perfect but not dense throughout any interval whatsoever. Erect at each point of this set a perpendicular of unit length, drawn in the direction of the positive axis of ordinates. The region T consists then of the points of the upper half plane that remain after these perpendiculars have been cut out. Thus the boundary of T will in this case have a positive content if the content of the assumed set of points on the axis of abscissas was positive. Moreover, there now exist points of the boundary that cannot be approached along a continuous curve, namely, any interior point of a perpendicular about which perpendiculars cluster from both sides.

In the present paper the existence of the Green's function for the most general simply connected region T is established by methods due to Poincaré, and thus it appears that the interior of any such region whatsoever can be transformed conformably on the interior of a circle.

Dr. Collins deduced the fundamental formulæ of spherical trigonometry by means of quaternions, employing the sine-cosine, in place of the scalar-vector notation.

The conception of a moving source treated in Professor McMahon's paper was suggested by the following problem set by a physicist who wished to use the solution in connection with an experiment: A source of heat (such as a candle) starts at a given point of a very long straight wire, and is moved with uniform velocity along the wire; it is required to find the temperature at any assigned point of the wire at any future time; and especially to compare the temperature gradient in front of the candle with that behind the candle, when these gradients have become practically stationary relatively to the moving source. The methods of Fourier are not directly applicable, but a solution can be obtained by using Kelvin's expression for the effect of an instantaneous source of heat placed at a given point of an infinite rod, and superposing the temperature effects due to a succession of such instantaneous sources placed at successive points of the line. The solutions for a finite rod with various terminal conditions are then given; and the corresponding problems for an infinite plane and for a finite rectangle are solved. The results can be extended to three dimensions.

Lagrange found particular solutions of the problem of three bodies of two types: (a) the three bodies lie always in a straight line; (b) the three bodies are always at the vertices of an equilateral triangle. These solutions still exist when one of the bodies becomes *infinitesimal*, and when the two finite bodies move in *circles*. The problem is to find whether solutions exist in which the infinitesimal body describes closed curves in the vicinity of the points defined by the Lagrangian solutions. In Dr. Moulton's discussion use is made of the methods employed by Poincaré in his prize memoir in the *Acta Mathematica*, volume 13. The results obtained have points of contact with Darwin's memoir in the *Acta Mathematica*, volume 21. The principal results are the following: Let $1 - \mu$ and μ represent the masses of the finite bodies. Let $\mu = \mu_0 + \lambda$. There is a periodic solution for every μ_0 in which the infinitesimal body moves in the plane of the motion of the finite bodies, in the vicinity of each one of the Lagrangian straight line solutions, and in which the coordinates are expansible as power series in λ, vanishing with λ, and with periodic coefficients, the periods depending upon μ_0. Thus there is an infinite number of orbits vanishing with $\mu = \mu_0$. In the case of the equilateral triangular points in the plane there are two infinities of orbits with different periods vanishing with $\mu = \mu_0$, *if* $\mu_0 < .3086$. In the vicinity of the equilateral triangular points in space of three dimensions there is an infinity of solutions, with periods differing from the other two, which vanish with $\mu = \mu_0$ for *every value of* μ_0.

Miss Grow's paper is in abstract as follows: The transformation

$$T: \quad \begin{aligned} z_1 &= \lambda_1 z_1' + \mu_1 z_2', \\ z_2 &= \lambda_2' z_1' + \mu_2 z_2', \end{aligned}$$

converts $f=e_z^n$ where the coefficients of f are independent, into the form

$$f'(z')=e_\lambda^n z_1'^n+\binom{n}{1}e_\lambda^{n-1}e_\mu z_1'^{n-1}z_2'+\cdots+e_\mu^n z_2'^n.$$

If it is possible to transform the quantic by T so that any two assigned coefficients, for example the $(n-r+1)$th and $(k+1)$th, shall vanish simultaneously, the necessary and sufficient condition is

$$e_\lambda^r e_\mu^{n-r}=0,\quad e_\lambda^{n-k}e_\mu^k=0$$

$$(r=1,2,\cdots,\ n-1;\ k=n-1,\ n-2,\cdots,1;\ k\neq n-r).$$

Consider $\begin{array}{ll}(1) & e_z^r e_\mu^{n-r},\\ (2) & e_z^{n-k}e_\mu^k\end{array}$ and $\begin{array}{ll}(3) & e_z^{n-r}e_\lambda^r,\\ (4) & e_z^k e_\lambda^{n-k}.\end{array}$

If (1) and (2) vanish simultaneously, their resultant $R_{n-r,k}$ vanishes; also if (3) and (4) vanish simultaneously, their resultant $R_{r,n-k}$ vanishes. $R_{n-r,k}$ and $R_{r,n-k}$ each have $[(n-r)(n-k)+rk]$ zeros in the plane; therefore (1) and (2) will vanish simultaneously when μ is a root of $R_{n-r,k}$, and (3) and (4) when λ is a root of $R_{r,n-k}$. f is always a factor of $R_{n-r,k}$ and $R_{r,n-k}$, or $R_{n-r,k}=fC_{n-r,k}$, and $R_{r,n-k}=fC_{r,n-k}$.

The z-points of (1) and (2) are the λ-points of $R_{r,n-k}$. Similarly the z-points of (3) and (4) are the μ-points of $R_{n-r,k}$, *i. e.*, the μ-points of $R_{n-r,k}$ and λ-points of $R_{r,n-k}$ are so related that

$$e_\lambda^{n-r}e_\mu^r=0,\qquad e_\lambda^k e_\mu^{n-k}=0.$$

When $r=k$, $C_{n-,k}$ and $C_{,n-k}$ coincide in $\overline{C}_r$. The roots of $\overline{C}_r$ go in pairs which have reciprocal relations. In the transformation T, let λ and μ be roots of $C_{r,n-k}$ and $C_{n-r,k}$ respectively and call this set of transformations $T_{n-r,k}$. Interchange λ and μ and call this set $T_{,n-k}$. The non-singular binary quantic can be linearly transformed by the $[(n-r)(n-k)+kr-n]$ transformations $T_{n-r,k}$ so that its $(n-r+1)$th and $(k+1)$th terms shall vanish, and by $T_{r,n-k}$ so that its $(n-k+1)$th and $(r+1)$th terms shall vanish. Also if λ and μ are zeros of $\overline{C}_r$ in corresponding pairs, we get two sets of transformations $\overline{T}_r$ and $\overline{T}_r'$ by means of which its $(r+1)$th and $(n-r+1)$th terms can be made to vanish in $2r(r-n)+n(n-1)$ different ways.

The theory is applied to the cubic, quartic, quintic, and sextic.

The following is a summary of Dr. Porter's paper : Consider $3m - s - 1$ arbitrary fixed points P on a non-singular cubic C_3, and $u_i = \int_{ab}^{x_i y_i} du$ the integral of first kind on C_3, (ab) being a point of inflexion. If an m-ic have a $s - 1$ order contact at u_1, it will cut C_3 again at u_2, $su_1 + u_2 \equiv C$ (mod. ω, ω') where $C = \Sigma u_i$ at the points P. The Schliessungsproblem thus suggested yields at once a proof of Fermat's theorem $a^n - a \equiv 0$ (mod. n (prime)) and the generalized form of the theorem $F(a, n) \equiv 0$ (mod. n). When $m = 1$, $s = 2$, we have systems of closed polygons. In case the polygon is a triangle, the equation of C_3 referred to it may be written

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 2\eta = 0.$$

The twenty-four in-circumscribed triangles thus determined fall into four groups, each associated with an inflexion triangle, and each triangle of a group six ways perspective with its associate inflexion triangle. This configuration of inflexion triangles and in-circumscribed triangles presents numerous interesting geometrical properties.

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THE UNDERGRADUATE MATHEMATICAL CURRICULUM.

REPORT OF THE DISCUSSION AT THE SEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE final session of the Seventh Summer Meeting of the Society was devoted to an organized discussion of the following question :

What courses in mathematics shall be offered to the student who desires to devote one-half, one-third, or one-fourth of his undergraduate time to preparation for graduate work in mathematics?

The following topics were also suggested as a general basis of discussion :

How early in the course may the lecture method be used with profit ?

How can the undergraduate best be trained to use the library?

How shall the history of the subject be presented?

Shall we try to place the calculus as early as possible?

Where shall we place the solid analytics? modern geometry? projective geometry?

Shall spherical trigonometry and geometric conics be given a place in the curriculum?

How much of the theory of equations may be presented with profit in the freshman year?

What courses in applied mathematics are needed as preparation for graduate work in pure mathematics?

Shall work in differential equations be merely a problem course, or shall it take up the theoretical side of the question? If the latter, to what extent?

At what points shall the work intended to fit the student for graduate work differ from that intended to fit him for secondary school teaching?

Are the best results in graduate work secured from students who have devoted most of their undergraduate time to mathematics, or from those who have combined a fair amount of mathematics with a more general culture?

Shall the undergraduate school attempt to attract the student to graduate work by offering elementary courses in the more advanced topics, or shall it confine itself to fundamental work in algebra, analytics, and calculus?

The principal papers are given in abstract below.

PROFESSOR MOORE.

If the student is to be properly prepared for graduate work the teacher must be himself familiar with modern mathematics; and the fundamental modern ideas and methods must give form to his work as an instructor of undergraduates. Thus the teacher should have clearly in mind the notions of the pure analysis—of the sequence of positive integers as a complete set, of the synthetic determination of all real numbers from this sequence of integers and of the complex variable as a double real variable. Similarly he should have the (usual) idea of (a) geometry as built upon a body of basal notions and axioms.

In general, mathematics should be regarded as divided into a number of distinct but closely related sciences. Each of these sciences consists of a body of sharply defined and inter-related ideas as foundation, with a superstructure consisting of results deductively secured from these fundamental ideas. For every science the question of or the condition for its existence as a body of non-conflicting statements is of prime importance, and especially by these existence considerations are the various sciences related and organized into higher sciences.

Intuition is by no means to be excluded from mathematical work. On the contrary, in every way let intuition be kept alive and active, but when a new idea is introduced as an intuition let it be sharply defined with the understanding that in so far as subsequent work depends on that intuition a new branch of mathematics is being originated.

In illustration: It is questionable whether the undergraduate course of a student who is not specializing extremely in mathematics may advisably contain the existence theory of the real irrational number in its ultimate form as a part of pure analysis. The student, however, may be led to grant the fundamental properties of the system of all real numbers as intuitively true for the system of all points of a straight line. On the basis of these assumptions the student develops a geometric analysis which will only later in pure analysis receive a deeper foundation, the character of which however should at once be briefly indicated. Similarly, in dealing with the complex variable in the Gauss plane, the radius and the angle may be regarded as simple ideas, although they are to appear as much more complicated creations in pure analysis.

PROFESSOR HARKNESS.

In these days of extreme specialization half the college time is too little to serve as an adequate basis for later work, unless the student enters college well prepared in elementary mathematics. With improved educational machinery in the secondary schools, there is no reason why a fair number of our college freshmen should not have had some preliminary training in analytic geometry, differential calculus, statics, and dynamics. This is the case in England and would be the case here, if the brighter boys were not held back in order that they may not get too far in advance of the less intelligent members of their classes.

As regards college students I urge the importance of increased attention to applied mathematics. In mathematics as in everything else there are fashions. At one time the subject of invariants, at another that of the theory of functions, and later still Lie's continuous groups have engrossed the attention of pure mathematicians; today algebraic numbers, or briefly those branches of mathematics which we associate with the names of Kronecker, Hilbert, Hensel, and Weber, are in a central position. All this time applied mathematics have been out of fashion, but there are many signs that a change is coming and the intelligent college instructor will do well to anticipate this change as far as possible.

However this may be, it is of great importance for the pure mathematician of today that he should be well grounded in the fundamental principles of statics, dynamics, and the theory of the potential. Riemann's programme cannot be appreciated at its proper value without a firm grasp of the fundamental principles of the theory of the potential so far as it relates to Dirichlet's Problem. It would be easy to multiply examples of this kind; for instance Schottky's memoir in the 83d volume of Crelle contains Weierstrass's Lückensatz, and is therefore of the greatest interest to the mathematician concerned with algebraic functions and Abelian integrals, but papers of this kind are apt to be passed by as unintelligible unless the reader has had some preliminary training in the discussion of conjugate functions, conform representation, and the behavior of the potential in multiply connected regions.

A good early training in statics and dynamics (including dynamics of a particle and the elements of rigid dynamics) would go far to remove many of the difficulties experienced by graduate students in the reading of modern memoirs, and would widen their mental horizon in a variety of ways. In the college instruction itself such a training would throw much needed light on many of the problems of the differential and integral calculus; partial differential equations would be connected with physical problems, and the undergraduate would feel, as he hardly feels at present, that the formulæ he meets with are capable of concrete representations.

I urge the necessity of a thorough revision of college courses in differential calculus. The subject is too often taught along traditional lines; the lecturer should ask himself whether certain parts of his subject possess the same importance today that they had in Euler's time. Much might be rejected as of secondary importance in comparison with the newer developments. Our text books abound, too, with proofs that are no proofs and with extremely misleading statements. It is very desirable that such proofs should be rejected, or given only with clear indications of their limitations. A graduate student in the theory of functions often finds it impossible to rid his mind of erroneous notions derived from the days when he first began the study of the differential calculus.

While due emphasis should be laid on the fundamental concepts of function, limit, continuity, etc., the work of the undergraduate should be lightened as far as possible. He should not be asked to differentiate $x^{x^{x}}$ or to work out examples which illustrate no general theory and are unlikely to be of use to him later on. Moreover care should be exercised when he comes to applications; the simple cases are usually discussed correctly, but the writers of our text books are too apt to go further and attempt a partial and inaccurate discussion of more difficult cases. Here again it would be easy to multiply examples; it will be sufficient to refer to the criteria for maximum and minimum values and to the accounts usually given of the higher singular points on an algebraic curve.

In conclusion I desire to emphasize the great importance of thoroughness; the greatest service that can be rendered to a prospective graduate student is to give him a complete mastery over those parts of the technique of mathematics which he is certain to need at a later stage of his work.

PROFESSOR OSGOOD.

In school instruction in geometry the idea is emphasized from the beginning that certain definitions and axioms are laid down, on which the whole subject of geometry is built up by means of logical reasoning; and the pupil's reasoning power and appreciation of geometric truth are developed by means of original exercises that are given him to work out by himself. That instruction in algebra is at present far less highly developed as regards effectiveness for the purposes both of general training and of giving the pupil a correct conception of the subject of algebra itself, is due doubtless in part to the formal side of the subject. Dexterity in the manipulation of formulas is indeed important for later work; but the foundation of algebra is arithmetic, and the arithmetic source of algebraic principles ought not to be lost sight of to any such extent as is at present the case. How far improvement here in school instruction is possible I will not undertake to say, but of this I am sure, that this difficulty must be met and can be met from the start in instruction in calculus. The foundation of the calculus as we have the science to-day is arithmetic. Now it is neither feasible nor desirable to begin a college course in calculus by first developing the number concept. I find it useful to assume the conception of a scale; more precisely, to make the assumption that, two points having been chosen at pleasure on a straight line to represent the numbers 0 and 1, every other (real) number can then be represented by a point of the line; and conversely, to every point of the line there will correspond a number. Similarly the conception of a function of a single variable or of two variables, is coextensive with the intuitive picture of a plane curve or of a surface. The fundamental principles of the calculus must be taught in a manner wholly different from that set forth in the text books if they are to become flesh and blood to the student. It is an easy method for the teacher to expound the notion of the limit in the opening lectures of the course and from there on to appeal to this conception, this mode of thinking, as if it had become a part of the mental machinery of the beginner. No greater mistake can be made. The student will indeed learn to perform differentiations and integrations; but he will not make progress in mastering the central ideas of the calculus. The notion must be set forth repeatedly throughout the whole course. A valuable aid in making this notion clear is furnished by the application of the calculus not merely to differential geometry, but (the integral as the limit of a sum having been introduced early in the first course in calculus) to problems in volumes, centers of gravity, moments of inertia, fluid pressures, gravitational attractions, etc.* It is the deduction of the formula that brings out anew each time the fundamental principle of the limit, while the problems themselves are typical applications of the calculus to physics, and interest and train alike the student who is preparing for work in pure mathematics and him who is to concern himself primarily with its applications.

I have thought it necessary to state with some fulness some of the leading features of a course in calculus, for there are still colleges in this country in which calculus means a systematic study of Williamson's text book—formal differentiations, formal integrations, and a total lack of the ideas that are central in the calculus of today. A course in calculus along the lines that I have indicated is effective in introducing the student to modern mathematics, and its notions and methods are in themselves interesting and valuable for the purpose of general training. It is best placed in the sophomore year for students who have entered college on the ordinary requirements.

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* This part of an introductory course in calculus is well set forth by the problems in Chap. V. of Professor Byerly's Problems in Differential Calculus.

As to how early the lecture method may be used with profit I may say that at Harvard the lecture method has been employed with success for over twenty years in practically all of the courses in mathematics, except in the freshman course in solid geometry. Written exercises, chiefly problems, are handed in by the student at each meeting of the course, and the instructor frequently spends some minutes on points that may have given general difficulty or require special comment.

It has been the practice at Harvard, ever since the graduate school came into existence, not to separate the graduate and the undergraduate departments as regards methods of instruction or the unrestrained admission of properly qualified students registered in the one department to the courses of the other department. Since there is free election from the freshman year on and the choice of courses is large after a student has passed beyond the elementary courses in calculus, geometry, and mechanics, it is the rule rather than the exception that the bright student, following his inclination to a special line of work in mathematics, will proceed rapidly to higher courses in this particular line; and thus a course ordinarily taken by first year graduates, say, will have seniors or even juniors in it, while on the other hand students that have paid more attention to other lines of work may not take it till a later graduate year. The advantage that is gained from this arrangement is obvious. The older students bring with them greater maturity for the work of the course and contribute to setting a higher standard, and the younger ones derive from the work they see their fellow students doing profit which they could not get from the lecture alone. In this respect the German university is especially strong. There, a student has free scientific intercourse with more advanced fellow students from the time he enters the university till he becomes himself a professor. At Göttingen, for example, there is a mathematical society composed of the university teachers, students whose attainments are measured approximately by the doctor's degree, and a goodly number of advanced students whose attainments are intermediate between those of these two classes. At its weekly meetings the current literature is discussed and lectures on current topics are given by its members. The formal meeting is followed by an informal one, usually held at an inn or in a summer garden, at which there is abundant opportunity for free discussion of scientific topics. The advantages that accrue to the undergraduate courses in one of our colleges from the presence of a graduate school are not unlike the benefits that the younger members of this society derive from their intercourse with its older members.

Professor Morley.

As to the socalled geometrical conics, a little of this might well be tacked on to elementary plane geometry. The beginner is apt to suppose that Euclid's methods in plane geometry are good only for the line and circle. Spherical trigonometry should form not a separate course but a chapter in solid analytics. The latter is somewhat neglected in college work, owing to the difficulty which poor students find in it. The plan of joining both two and three dimensions in one book is a good one, but the books in use are too old fashioned for the class of men we are considering. As to the theory of equations in the old sense (that of Burnside and Panton), it can be and I think should be attached to elementary analytic geometry. For instance when the curve

$$y = a + bx + cx^2$$

is mastered and its slope can be found, then the curve

$$y = a + bx + cx^2 + dx^3$$

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should be fully discussed. The points where the curve meets $y = 0$ would lead to the consideration of Horner's method. The general parabolic curve would then be mentioned. Taylor's theorem would make its first appearance here, as a convenient formula for changing the origin and discussing the curve near a point other than $x = 0$. Similarly the notion of covariants should grow out of geometry.

In general the notions of arithmetic and geometry with which a student enters college should be developed by interaction, say through plane and solid analytics and a course in calculus, before either the theory of arithmetic or that of geometry is considered philosophically. This is in effect a plea for the existing order of subjects, if they are presented in a way which leads up to the subjects as they stand, and satisfies both the refined views of the teacher and the common sense of the student. Thus, in the case of the calculus, the books used to begin with some tremendous generalities.

We have either to justify these, or to throw them aside, or to do both. We should do both—first have a calculus proper, of the kind dear to physicists, accurate but not abstract, carefully weeded of difficult generalities ; and at a later stage a philosophical theory. To begin with the latter would to my mind be a mistake ; the appeal would be to too limited a class.

Thus in my view the first half of a college course would be the same for the future mathematician as for the future physicist. And in fact there would be direct gain for the mathematician if the work were better adapted for the physicist than at present. Thus we ought not only to connect arithmetic with geometry, but also with dynamics, and perhaps with other physical subjects such as electricity, taking care that the student has the definite physical concepts to which we are appealing.

Lastly we must take hints from the technical schools, for instance with regard to drawing diagrams and the use of integrators or of abaci.

PROFESSOR YOUNG.

Considering the fundamental topic of this discussion to be:—*The work in mathematics in college: its purpose, scope, methods and relation to graduate work in mathematics*, my remarks are to be made from a special point of view, viz., when the purpose is to prepare teachers of mathematics for the secondary schools.

I. *The need of such preparation.*—It is a truism to say to this audience that secondary school teachers of mathematics should have had a collegiate education with special attention to mathematics either by electives in the college course or by some graduate study. It is equally well known to this audience that this standard is by no means generally reached. The time seems to be ripe, however, for setting up the general culture of a full college course and the special preparation of a certain minimum of mathematical attainment as norm for the educational qualifications of new appointees to the work of teaching mathematics in our secondary schools.*

II. *The preparation, when to be attained.*—However desirable some graduate work may be, the fact must be recognized that at present the norm mentioned above represents a decided advance, and that for some time to come the majority of the secondary school teachers will have no preparation beyond a college course. The college program should consequently include all the work needed for the minimum of preparation.

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* Cf. Resolution III of the Committee on the College Entrance Requirements, p. 30 of report submitted July, 1899.

III. *The subject matter of the minimum course.*—As the minimum of mathematical attainment the following courses are suggested:—Thorough courses in plane trigonometry and college algebra, and good first courses in analytic geometry of the conic sections, differential and integral calculus, theory of algebraic equations, determinants, modern synthetic geometry, and, if possible, analytic mechanics.

IV. *Pedagogic preparation.*—While experience has shown that special training in the art of teaching mathematics is of high value, it is desirable that the recipient of this training be as mature as is feasible. The best place for the pedagogic training and instruction is therefore *after* the college course; either in the university or by some form of apprenticeship in the secondary schools. (Under favorable circumstances, helpful work of this sort could doubtless also be done in the senior year of the college.)

V. *Mathematical Independence.*—By mathematical independence I mean a degree of mastery of subject matter and methods such that the belief in the truth of results is based solely on the authority of one's own reason, and such that one is conscious of the power to apply these results appropriately and correctly. Within a narrow range, this independence may be attained very early; even the young pupil may have it, widening its scope as his attainments increase. Indeed, good teaching strives to cultivate this independence from the outset, and the teacher who is to cultivate it must himself evidently possess it and in a distinctly higher grade. The pupil, the teacher, the scholar, each should be in his degree master of his work. The teacher's mastery requires that he teach on his own authority, and not on that of others (though the mastery of the teacher need not be in the same degree productive as the mastery of the scholar). Unfortunately, our teachers have only too often not attained this independence. Too frequently a text is taught rather than the subject, the teacher having in reality not gone beyond the pupil's stage of mastery.

It is easier to point out the difficulties than to specify the remedy. The attainment of some measure of scientific independence suggests itself. This is the German plan, but it is not now feasible here. It seems to me, however, that somewhat can be done even within the limits of the college course. Our present methods and programmes tend in one respect to encourage diffidence rather than independence.

Mathematics requires to be digested and assimilated as well as acquired. The trend of our work is to cover as many subjects and as much ground as possible, and this continual advance with its successive generalizations and its revelations of incompleteness in what has been accepted as complete tends to cultivate a lack of confidence, accentuated if the student is embarrassed by difficulties of detail. This feeling of uncertainty takes shape in reliance upon others (book or person) to pronounce final judgment upon whatever is done. Perhaps something could be done to give the student the more secure grip on the secondary school subjects which the teacher must have by taking up these subjects again at the close of the college course and treating their principles and methods (not new details, as a rule), with the greater thoroughness and breadth of view which are made possible by the collegiate mathematics as outlined in the minimum course above.

VI. *Difference between the work of the prospective teacher and the prospective graduate student of mathematics.*—In the college this difference may be very slight.

VII. The teachers needed (in college) to carry out the above programme fully must have the wide horizon of a university training.

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ON A MEMOIR BY RICCARDO DE PAOLIS.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

ABOUT twenty years ago de Paolis published a series of memoirs* dealing with the (2, 1) transformation of the plane; of these the second and third are concerned with applications of the theory to non-euclidean geometry and to quartic curves, while the first treats of the general theory of the (2, 1) transformation. De Paolis employs no analysis, all his results are obtained simply by geometrical considerations. The curves of the first plane (piano semplice) which correspond to the straight lines of the second plane (piano doppio) have only two free intersections, all the other common points being absorbed by the fixed (or fundamental) points by means of which the conditions to which the curves are subject are imposed. This, taking up a question which arises in connection with a memoir by Clebsch,[*] is the next in order to Cremona's memoirs,[†] the (2, 1) transformation naturally following the (1, 1). This very striking memoir, being already classical, is a fair subject for discussion; and in the following pages I propose to consider,[‡] in particular, de Paolis's classification of fundamental points and lines, which in my opinion he has based on symptoms of differences instead of on the real causes. When the (2, 1) transformation is regarded as but one case in the general theory of ($\varkappa$, 1) transformation, some of the special properties—those, in fact, to which its peculiar interest is largely due—take on a rather different aspect.

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* "Le trasformazioni piane doppie"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 1 (1877); pp. 511–544.

"La trasformazione piana doppia di secondo ordine, e la sua applicazione alla geometria non euclidea"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 2 (1878); pp. 31–50.

"La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine"; *Atti d. r. Accad. d. Lincei*, series 3ª, vol. 2 (1878); pp. 851–878.

The relation connecting the dimensions k of a system of curves of genus p with the number $\varkappa$ of variable intersections, is known to be

$$\varkappa - (k - 1) = p,$$

if the fundamental points are entirely arbitrary in position, and consequently impose independent conditions;

$$\varkappa - (k - 1) \leq p - 1,$$

if the points are connected in such a way that the conditions imposed are not all independent. This shows that for the (2, 1) transformation with arbitrary fundamental points,

$$p + k = 3.$$

Hence, as shown by de Paolis, there are two cases to consider, namely,

$$p = 1,\ k = 2; \qquad p = 0,\ k = 3.$$

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[*] Clebsch, "Ueber den Zusammenhang einer Classe von Flächenabbildungen mit der Zweitheilung der Abel'schen Functionen"; *Math. Annalen*, vol. 3 (1871).

[†] Cremona, "Sulle trasformazioni geometriche delle figure piane"; *Memorie d. Accad. d. Bologna*, series 2ª, vol. 2 (1863); vol. 5 (1865).

[‡] I have not attempted to keep to the lettering of the memoir, preferring rather to use that adopted in other places.

In the first of these the transformation net is composed of all curves that satisfy the assigned conditions; in the second, these curves form a three-fold infinity, from which the net is selected by means of three determining curves. Now so far as the determination by fundamental points is concerned, these three curves may be any three independent ones of the system that have not any additional points in common. The same thing holds whatever value x may have; the system of curves determined by the fundamental points is of dimensions k, and to obtain a transformation net we virtually impose $k-2$ extrinsic conditions, these being implied in the choice of the three determining curves. The theory of linear systems contained in a system of the same order, but a greater number of dimensions, has received very little special attention, owing to the fact that the theorems relating to such a system are simply deducible from those for the complete system; but there are one or two points in the theory that deserve to be remarked, and they are the ones that seem to me to throw some light on the true nature of the differences that de Paolis has attempted to formulate in dividing fundamental points, as also fundamental curves, into three species.

Let the curves, φ, of the first plane that correspond to the straight lines of the second plane be given as of order σ, having fixed multiple points of orders ρ_1, $\rho_2, \cdots$, at the fundamental points. There may be also fundamental curves. A curve β, of order ω, with multiple points of orders τ_1, $\tau_2 \cdots$ at the fundamental points, is called fundamental if it meets the curves of the given system only at the fixed points, that is, if

$$\sum \rho_i \tau_i = \sigma\omega.$$

These curves have been minutely studied by various Italian writers,* who have shown that any such curve is in general rational, and precisely determined by the fundamental points through which it passes, facts which lead to the equations

$$\sum\tau_i^2 = \omega^2 + 1, \quad \sum\tau_i = 3\omega - 1 \qquad \text{(Bertini)}.$$

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* Caporali, "Sopra i sistemi lineari triplamente infiniti di curve algebriche piane"; *Collect. Mat. In Mem. Dom Chelini*, 1879.

Jung, "Sui sistemi lineari di curve algebriche di genere qualunque"; *Rend. del R Ist. Lomb.*, series 2ᵃ, vol. 20 (1887).

Bertini, "Sulle curve fondamentali dei sistemi lineari di curve piane algebriche"; *Rend. del Circ. Mat. di Palermo*, vol. 3 (1889).

Castelnuovo, "Ricerche generali sopra i sistemi lineari di curve piane"; *Mem della R. Accad. di Sc. di Torino*, series 2ᵃ, vol. 42 (1891).

If however the fundamental points are specialized in such a manner that the conditions actually imposed on the curves φ are fewer, by e, than the conditions apparently imposed (that is, if the fundamental points have an *excess** e for the curves φ, or if the system of curves φ is of *sovrabbondanza*† e), then β need not be rational; its genus may have any value that does not exceed e (Castelnuovo). Any irreducible fundamental curve β is a factor in k independent curves of the system, that is, in a contained $(k-1)$-fold system; and there is the general theorem that β will be a *non-repeated* factor in the generic curve of this $(k-1)$-fold system.

As regards the fundamental points and curves, two questions call for investigation. The first relates to the possibility of a curve with a multiple point of order higher than the stated value ρ at a fundamental point, or, we may say, to the existence of a φ that shall be irregular at one or more of the fundamental points; the second to the possibility of a curve with β as a repeated factor, without any specialization in the position of the fundamental points. These two questions I have considered in another place;‡ hence the results only need be mentioned here.

On the first question the conclusion is that the k-fold system of curves contains a curve that is irregular at any fundamental point whose order is less than k (or, by an extension of the argument, at any m fundamental points the sum of whose orders does not exceed $k-m$); but in selecting from the complete system three curves for the determination of a net, no two must be chosen that are irregular at the same point, for such a choice would introduce new fundamental points indefinitely near to the stated ones. Irregularity on one curve at any fundamental point does not of itself imply any specialization in the fundamental points.

As to the second question, the result is that the $k+1$ determining curves of the system can be chosen as the first $k+1$ of the following series:—

one, that does not contain β,

two, that contain β but neither contain nor produce β^2 (*i. e.* when taken in any linear combination),

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*Macaulay, "Point groups in relation to curves," *Proc. Lond. Math. Soc.*, vol. 26 (1895).

†Castelnuovo, loc. cit.

‡"Note on linear systems of curves," *Nieuw Archief voor Wiskunde*, Tweede Reeks, Derde Deel, 1898; also BULLETIN, vol. 4 (1897–8); p. 88.

three, that contain β^2, but neither contain nor produce β^3,

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$h+1$, that contain β^h, but neither contain nor produce β^{h+1}, where every $\rho \geqq h$ times the corresponding τ, and at least one $\rho < (h+1)$ times the corresponding τ.

If however $k+1 > \frac{1}{2}(h+1)(h+2)$, there are also curves, $k+1-\frac{1}{2}(h+1)(h+2)$ in number, containing higher powers of β; the numbers containing successive powers being N_{k+1}, N_{k+2}, etc., no general formula for an N can be given, all that can be said is that the successive N's do not increase. As these last contain β to a power so high that $h\tau > \rho$ at some one point, the curves are irregular. Hence it is convenient to distinguish the two parts of the series as regular and irregular. It is at once evident that any one of the curves can be modified by the addition of multiples of later members of the series without affecting its relation to β.

If $k = 2$, the curves by which the net is determined are necessarily the first three, that is,

$$\varphi_1, \quad \beta\psi_2, \quad \beta\psi_3.$$

If $k > 2$, any three may be chosen that have not any fixed points in common other than those expressly provided for at the fundamental points. One must obviously be φ_1; of the other two, one may belong to the irregular part of the series, but not both, inasmuch as this would cause irregularity at one point on both curves.

For example, the 12-fold system of 15-ics, with multiple points of orders 8, 6, 6, 5, 4, 4, 4 has the cubic 2, 1, 1, 1, 1, 1, 1 as a fundamental curve; and $h = 4$. The complete system can therefore be written as follows, where all thirteen curves are regular:

$$\varphi_1, \quad \beta(\psi_1, \psi_2), \quad \beta^2(\chi_1, \chi_2, \chi_3), \quad \beta^3(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4), \quad \beta^4(\zeta_1, \zeta_2, \zeta_3).$$

The system residual to β^4 is the net of cubics with fixed points of orders 0, 2, 2, 1, 0, 0, 0, which is simply the straight line $A_2A_3 \times$ the net of conics through $A_2A_3A_4$. Hence in this case we must not choose, for the determination of a net, two curves containing β^4, even though both would be regular, for the net thus determined would have other fixed points, namely, the three points in which A_2A_3 meets the 15-ic φ_1. But with the 12-fold system of 14-ics with points of order 8, 6, 4, 4, 4, 4, 4, which has the same fundamental curve, there is not this restriction on freedom of choice.

We can at once write down a system with any possible fundamental curve, and with any arbitrary residual system, by means of the theorem :—If a curve β of order ω, rational and precisely determined by fixed points of order $\tau_1, \tau_2, \cdots$, which impose independent conditions, has h free intersections with the generic curve of a q-fold system of n-ics conditioned solely by its behavior at the fixed points, having these as multiple points of orders r_1, $r_2, \cdots$, then β is a fundamental curve of the system of $(n + h\omega)$-ics determined by multiple points of orders $r_1 + h\tau_1$, $r_2 + h\tau_2, \cdots$, at the fixed points.

The fundamental curve is not possible unless the numbers expressing the orders of its multiple points satisfy the relations

$$\tfrac{1}{2}\,\Sigma\tau(\tau + 1) = \tfrac{1}{2}\,\omega(\omega + 3) - \delta,$$

$$\tfrac{1}{2}\,\Sigma\tau(\tau - 1) = \tfrac{1}{2}(\omega - 1)\,(\omega - 2),$$

and then δ simple points must be taken on β to complete its determination.

The proof of the theorem is obvious. The number of free intersections of β with the generic curve of the system is

$$\begin{aligned}\omega(n + h\omega) - \Sigma\tau(r + h\tau)&\\ &= n\omega - \Sigma r\tau + h\,(\omega^2 - \Sigma\tau^2)\\ &= h\,[1 + \omega^2 - \Sigma\tau^2] = 0.\end{aligned}$$

Hence β is either an h-fold factor or a fundamental curve ; but as the dimensions of the system

$$\begin{aligned}&\geqq \tfrac{1}{2}(n + h\omega)\,(n + h\omega + 3) - \tfrac{1}{2}.\Sigma(r + h\tau)\,(r + 1 + h\tau)\\ &\geqq q + \tfrac{1}{2}\,h\,(h + 1) > q,\end{aligned}$$

it follows that the system of $(n + h\,\omega)$-ics is more extensive than the given system of n-ics, hence it is not simply this system with an extra factor β^h, but a more extensive system with β as a fundamental curve.

By means of this theorem we can write down a system with any desired fundamental curve, choosing a residual system at random. For example, if the fundamental curve be a rational cubic, we must use also for its complete determination six simple points on the curve, which can then be expresssed as $2, 1^6$. If the residual system is to be *all* conics, then $h = 2 \times 3 = 6$; hence the desired system consists of 20-ics $(6 \times 3 + 2)$, with one multiple point of order 12, and six sextuple points.

In the general rational transformation, in which the coördinates of a point in the second plane are given rationally in terms of the coördinates of a point in the first plane by equations of the form

$$\varphi_1 : \varphi_2 : \varphi_3 = x_1' : x_2' : x_3',$$

to a single point P in the first plane there corresponds a single point P' in the second plane; but to this one point in the second plane there correspond in the first plane all the $\varkappa$ points P_1, $P_2, \cdots$, P_κ which are the intersections of the pencil of φ's through P. If however P is a fundamental point of order ρ, *e. g.*, the point (0, 0, 1), there is no longer any unique correspondent P'. The φ's being arranged by powers of x_3, the equations become

$$\frac{x_1'}{u_\rho x_3^{\sigma-\rho} + u_{\rho+1} x_3^{\sigma-\rho-1} + \cdots} = \frac{x_2'}{v_\rho x_3^{\sigma-\rho} + \cdots} = \frac{x_3'}{w_\rho x_3^{\sigma-\rho} + \cdots};$$

the correspondent to the actual point (0, 0, 1) is absolutely indeterminate, for all these denominators are zero; but taking a point indefinitely near to (0, 0, 1), in a definite direction given by $x_1 = \lambda x_2$, the corresponding point is determinate, the equations now becoming

$$\frac{x_1'}{u_\rho(\lambda, 1)} = \frac{x_2'}{v_\rho(\lambda, 1)} = \frac{x_3'}{w_\rho(\lambda, 1)}.$$

Taking in succession points close to (0, 0, 1) in all directions, that is, allowing P to describe an indefinitely small simple circuit about (0, 0, 1), P' describes a curve α', rational and of order ρ, inasmuch as the x_1', x_2', x_3' are rationally expressed in terms of the parameter λ by these equations. (See Clebsch-Lindemann. and other places.) Returning to the first plane, the complete correspondent to α' is a curve α_i of order $\sigma\rho_i$, which can be shown* to have at A_j a multiple point of order $\rho_i\rho_j$, but at A_i one of order $\rho_i^2 + 1$. This curve α_i is the complement of the fundamental point, the locus of the $\varkappa - 1$ points associated with a point that describes a circuit about A_i. If $\rho_i = 1$, it is simply the particular φ that has a double point at A_i.

Thus while any ordinary point of the first plane corresponds to a point of the second plane, and is associated with $\varkappa - 1$ points of the first plane, there are exceptional points, namely, the fundamental points; any one of these in

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*C. A. Scott, "Studies in the transformation of plane algebraic curves"; *Quar. Jour. of Math.*, vol. 29 (1898).

(iii) lies at A_1, which is then a fundamental point of the third species. Such a point arises, as has just been shown, from irregularity on one φ; it can arise also from another cause, the presence of what de Paolis calls a double point, to be explained later.

For a non-specialized transformation system with $\kappa = 2$, $p + k = 3$. If $p = 1$, $k = 2$; the only "irregularity" possible on a φ is the existence of a double point where the stated conditions impose only a simple point, and it has already been pointed out that this cannot properly be regarded as an irregularity. If $p = 0$, $k = 3$; we cannot produce irregularity on one φ at two points, for this requires the imposition of four conditions, and is therefore possible in a system only if the generic curve has as many as four degrees of freedom. Hence the fundamental points of the second species do not present themselves unless the determining points of the system are specialized, in which case $p + k > 3$. The fundamental points of the third species can occur, for $k = 3$ allows us to raise a double point to a triple point, and then an extra intersection lies at A_1, A_1 is associated with itself; there is no complementary curve, for the two points of a group are fully accounted for.

This classification of fundamental points lends itself to the case $\kappa = 2$, the characteristics of the three species being then mutually exclusive, inasmuch as the one point associated with any point P, near to A_1. cannot satisfy more than one of the three conditions. But if $\kappa > 2$, *e. g.*, $= 4$, and k is sufficiently great, we may have the three points complementary to P exhibiting all three characteristics. As P describes an indefinitely small simple circuit about A_1, P_1 may describe a curve, P_2 may lie at A_1, P_3 at A_1, and then the curve described by P_1 will be simply the irregular φ, φ_2. Thus de Paolis' classification is based on properties which are insufficient for purposes of discrimination when $\kappa > 2$.

Two points of a general κ group (the intersections of a pencil of φ's), can come together only if the φ's have contact; the locus of such points is therefore the Jacobian of $\varphi_1, \varphi_2, \varphi_3$. On forming this, in certain circumstances factors will present themselves, possibly to be rejected in finding the locus of points of true contact. De Paolis calls this locus the "curva doppia"; it is the Jacobian reduced by the rejection of irrelevant factors. As a point P_1, together with its associate P_2, describes the Jacobian, the correspondent P' describes a curve in the second plane, the "curva limita" of de Paolis [synoptic curve, *Q. J.*, loc. cit.]. In the general theory, with $\kappa > 2$, we have to take into account also the co-jacobian, the locus of the remaining points P_3, $P_4, \cdots, P_\kappa$; a detailed study of the relations of these curves, and their use in transformation, is to be found in the article in the *Quarterly Journal* already referred to.

Examining the equations of the Jacobian, it is seen to represent a curve of order $3(\sigma - 1)$, with multiple points of order $3\rho - 1$. If however one φ is irregular at A_1, having there a multiple point of order ρ', the order of this point on the Jacobian is $\rho' + 2\rho - 2$; the terms of this order are affected with a numerical factor $\rho' - \rho$. If now $\rho' = \rho + 1$, this has the normal value $3\rho - 1$; but if $\rho' \geq \rho + 2$, then $\rho' + 2\rho - 2 \geq 3\rho$. Thus the Jacobian is not affected by the simplest type of irregularity, that in which the order of a fundamental ρ-point is augmented by unity on one φ. With the restriction on the scope of the investigation adopted by de Paolis, namely, $\kappa = 2$, it is not possible to have $\rho' = \rho + 2$, hence this irregularity on the Jacobian is not mentioned. The irregularity that is considered on p. 523 has a different origin; it is due to the presence of a double point, whose nature is considered separately.

Corresponding to the fundamental points in the first plane there are fundamental curves in the second plane, which de Paolis classes as of the first, second, or third kind, according to the nature of the fundamental points from which they are derived. He makes the statement, as regards those of the second and third kind, that they meet the curves of the second plane which correspond to the straight lines of the first plane in fixed points only. This is not the case if $\kappa > 2$; but possibly it is true for $\kappa = 2$, as stated by de Paolis, and not only for the curves that correspond to straight lines, but also for the correspondents to any curves in the first plane, though I must confess I find de Paolis' argument somewhat obscure.

If there be in the first plane a fundamental curve β, the equations of transformation are

$$\varphi_1 : \beta^{h_2}\psi_2 : \beta^{h_3}\psi_3 = x_1' : x_2' : x_3'.$$

Corresponding to any point on β we have in the second plane the point $B'(1, 0, 0)$; to the point B' there correspond all the intersections of $\beta^{h_2}\psi_2$, $\beta^{h_3}\psi_3$ that do not lie at the fundamental points, $\kappa - h_2h_3$ in number. Hence the fundamental curve β replaces h_2h_3 of a group of κ points, thus displaying another exception to the general laws of the transformation. Of course h_2h_3 is necessarily less than κ. If $h_2 = h_3 = h$, to any point in the immediate neighborhood of B', in the direction $x_2' = \lambda x_3'$, there correspond h points on β, inasmuch as $\beta = 0$ meets $\psi_2 - \lambda\psi_3 = 0$ in precisely h points; thus if $h = 1$, that is, if β is a fundamental curve of the simplest type, as a point P' describes an indefinitely small circuit about B' one of the corresponding points describes the curve β; the relation of B' to β is precisely analogous to the relation of A to α'. This however does not hold if h is greater than unity.

If $h_2 < h_3$, to any point in the neighborhood of B' there correspond h_2 fixed points on β, determined by $\psi_2 = 0$.

Thus if $h_2h_3 > 1$, one or more of the correspondents to β is β itself, the remainder being a set of $\kappa - h_2h_3$ points.

If however ψ_2, ψ_3 have a common factor γ, we have in the simplest case the equations

$$\varphi_1 : \beta\gamma\vartheta_2 : \beta\gamma\vartheta_3 = x_1' : x_2' : x_3'.$$

To the point B' there now corresponds the reducible curve $\beta\gamma$, which is "monovalent" as regards the system of φ's; that is to say, if the φ is made to pass through one point on either component, it contains $\beta\gamma$ as a factor, the whole curve $\beta\gamma$ is equivalent to a single point as means of imposing conditions on the generic curve of the system of φ's. (Castelnuovo, loc. cit.) With any point on β there are associated the $\kappa - 2$ intersections* of ϑ_2, ϑ_3, and the curve γ; this curve is therefore a part of the complement of β.

If finally the equations of transformation are

$$\varphi_1 : \beta^{h_2}\gamma^{k_2}\vartheta_2 : \beta^{h_3}\gamma^{k_3}\vartheta_3 = x_1' : x_2' : x_3',$$

then a group of κ associated elements, of which β forms a part, contains in the simplest case $\kappa - h_2h_3 - k_2k_3$ points, not on $\beta\gamma$, together with β, h_2h_3 times, and γ, k_2k_3 times. If for instance $\kappa = 4$, and the transformation net is

$$\varphi_1, \quad \beta\gamma\vartheta_2, \quad \beta^2\gamma\vartheta_3,$$

then a set of four associated elements one of which is β consists of β, twice, γ, once, and one point. De Paolis divides the fundamental curves in the first plane into three kinds, according as the single associate (the only one if $\kappa = 2$) is a point, another curve, or the curve itself; but here again, just as in the case of the fundamental points, if $\kappa > 2$ the characteristics may coexist in any combination, hence the classification cannot profitably be placed on this basis in general.

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* There are $\kappa - 2$ in the simplest case that can arise, but they may be fewer in number.

On p. 518 de Paolis asserts that while fundamental curves of the first and second species are rational, those of the third species are *hyperelliptic*. Surely he has here been misled by the (2, 1) correspondence of the points of such a a curve with directions proceeding from B'. Any fundamental curve β, of whatever species, is subject to the conditions formulated in the memoirs of Bertini and Castelnuovo, conditions which are a direct consequence of the fact that all the intersections of β with the generic φ are fixed. If the complete system from which the φ-net is derived by selection has no "sovrabbondanza," then β is rational, since its genus cannot exceed the sovrabbondanza of the system. For a specialized choice of fundamental points, a fundamental curve of any species may cease to be rational.

From the ordinary determinant expression for the Jacobian it is at once seen that a fundamental curve that is an (h_2, h_3) factor presents itself as an $(h_2 + h_3 - 1)$-fold factor in the Jacobian; rejecting it, the residual part, J_0, is de Paolis' "curva doppia." This cuts β in $h_2 + h_3 - 2$ points; if $h_2 = h_3 = h$, these $2(h-1)$ points are $h-1$ points of simple contact. These are the general facts; de Paolis naturally notices only the cases that can occur for $x = 2$. If β is a simple fundamental curve, then β is a simple factor to be rejected from the Jacobian, and the reduced Jacobian does not meet β; if however β is a fundamental curve of the third species ($h_2 = 1$, $h_3 = 2$), the factor β^2 is to be rejected, and the reduced Jacobian meets β in one point.

A point on the "curva doppia" is a point of the plane where two complementary points come together, that is, are consecutive; the φ's through such a point have simple contact, the common tangent being the line determined by these consecutive points, and one φ has a double point. If there is an actual *coincidence* of two complementary points, de Paolis characterizes it as a double point of the plane, stating that the continuous system of points of the first plane may contain a discrete system of points associated with themselves. Any two φ's through such a point have two intersections, not due to contact. If the point is not fundamental, and $x = 2$, this can only be by means of a common factor, which is consequently a fundamental curve. The net is then determined by

$$\varphi_1, \quad \beta\psi_2, \quad \beta\psi_3,$$

but as ψ_2, ψ_3 are to have a common point on β, which fixes the sole (hitherto) free intersection of ψ_2 and ψ_3, it follows that ψ_3 is of the form $\psi'_2 + \beta\vartheta_3$; hence the net is determined by

$$\varphi_1, \quad \beta\psi'_2, \quad \beta^2\vartheta_3,$$

and the fundamental curve β is therefore of the third species, as de Paolis shows in a different manner (p. 524). The double point is the one point in which β and ψ'_2 intersect; and as any φ that contains β is of the form $\beta\psi'_2 + \lambda\beta^2\vartheta_3$, that is, $\beta\{\psi'_2 + \lambda\beta\vartheta_3\}$; every such φ has necessarily a double point at this point.

De Paolis takes the view that all such double points are isolated from the Jacobian, because he found them on the fundamental curves, and these he rejected in forming the "curva doppia." If however $\kappa > 2$, all the φ's through a point may have a double point there, without degeneration, that is, without the assistance of fundamental curves; this fact is outside de Paolis' argument, but it is essential to a full understanding. In this case the reduced Jacobian, the "curva doppia," has a double point. Such a point exists, for instance,* at (1, 0, 0) for the system φ_1, φ_2, φ_3, where

$$\begin{aligned}
\varphi_1 &= x_1(x_1 + x_2 + x_3)^2, \\
\varphi_2 &= 2x_1x_2^2 + (x_2 - x_3)(x_2 + x_3)^2, \\
\varphi_3 &= 2x_1x_3^2 - (x_2 - x_3)(x_2 + x_3)^2,
\end{aligned}$$

whose reduced Jacobian is

$$x_1(x_2 + x_3)^2 - (x_1 + x_2 + x_3)(x_2 - x_3)^2 = 0,$$

that is,

$$4x_1x_2x_3 - (x_2 + x_3)(x_2 - x_3)^2 = 0.$$

This point is of the same nature as the double point of which de Paolis speaks, but of higher order.

These are the only two ways in which all φ's of a pencil can have more than one intersection at a point that is not fundamental, independently of contact; either any two, and therefore all, have a common factor; or any two, and therefore all, have a multiple point.

The question whether a fundamental curve, which is a repeated factor in one or more φ's, shall or shall not be counted as a part of the reduced Jacobian, is after all a mere matter of definition; we may adopt any convention. A repeated factor (h-fold) in one φ, that is not a fundamental curve, presents itself as an (h—1)-fold factor in the

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* *Quar. Jour. of Math.*, vol. 29 (1898), pp. 357 and 379.

Jacobian, and has to be counted as an integral part of this curve; whereas a simple factor has no special relation to it. This lends some support to the view, to which I incline, that a fundamental curve which occurs as a repeated factor ought to be looked upon as a part of the true Jacobian; and bearing in mind the fact just proved as regards multiple points of the plane, it appears that this convention gives the most consistent set of results.

Again, a double point may be at a fundamental point A; a point complementary to A is then at A, which is therefore a fundamental point of the third species. This is the second manner of specializing the fundamental point, referred to above. It is the only one that de Paolis considers in any detail; and his purely geometrical treatment, while it has led him to a correct formulation of the properties of such a point, is not very conclusive as to its nature, which however is extremely interesting. This is considered analytically in the *Quarterly Journal of Mathematics*;* as the results of the investigation throw light on a rather obscure part of the memoir of de Paolis, it appears suitable to give here a résumé.

The pencil formed by the φ's that proceed in any given direction from a fundamental point of order ρ, contains one special φ, with a cuspidal branch whose tangent lies in that direction. If the direction is one of the $3\rho - 1$ indicated by the tangents to the Jacobian, the special φ has not a simple cuspidal branch, but a branch with a higher singularity, met by its own tangent in four points. In the simplest case, this can be resolved into two simple branches in contact, and the φ might be referred to as tacnodal; but to allow for the more complicated singularity, the term *extranodal* seems more convenient, as it is applicable to all cases. (Analytically, the repeated factor in the terms of degree ρ is a factor in the terms of degree $\rho + 1$.) These extranodal φ's are determined by the $3\rho - 1$ tangents to the Jacobian; the fundamental point may, however, be specialized in such a manner that for *every* direction the special φ is extranodal instead of cuspidal. This happens necessarily for all directions if it happens for some one direction in addition to the $3\rho - 1$; and the Jacobian has then a point of order 3ρ, while the synoptic curve contains a' as a factor. The number of intersections at A of two φ's proceeding in any assigned direction AP is in general $\rho^2 + 1$, but in this case it is $\rho^2 + 2$; of these ρ^2 are the fixed intersections, which

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*In the continuation of the article, already referred to, appearing in the current volume of the *Quar. Jour. of Math.*

are not enumerated in any $\varkappa$-group; thus two points of a $\varkappa$-group fall at P, in whatever direction P is taken. This may perhaps be most simply expressed by saying that the complement to A contains an evanescent circuit about A. This is what de Paolis refers to as a "punto fondamentale doppio"; it is one variety of the fundamental point of the third species.

The fundamental points and lines, the double points, and the Jacobian, in the first plane, with their correspondents in the second plane, form a kind of framework for the transformation, indicating everything exceptional; and a complete knowledge of them is essential to a full comprehension of the possible effects of the general rational transformation. If $\varkappa > 2$, the co-jacobian has to be taken into account also, and as I have here shown, the different species of fundamental points and lines can no longer be discriminated; hence the purely geometrical treatment becomes very involved, and it appears necessary to resort to analysis. Thus it is hardly to be expected that the treatment of transformation can be carried any further on the lines of Cremona and de Paolis; but the intrinsic interest of de Paolis' work is surely excuse enough for devoting some little space to it in a periodical of a critical and historical nature.

BRYN MAWR COLLEGE,
May, 1900.

NOTES.

THE Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will meet in the Ryerson physical laboratory of the University of Chicago, on Thursday and Friday, December 27th and 28th next. Titles and abstracts of papers to be read at this meeting should be in the hands of the Secretary of the Section not later than December 5th.

THE third (July) number of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Wave propagation over non-uniform conductors," by M. I. PUPIN; "Ueber Systeme von Differentialgleichungen denen vierfach periodische Functionen Genüge leisten," by M. KRAUSE; "On linear criteria for the determination of the radius of convergence of a power series," by E. B. VAN VLECK; "On the existence of the Green's function for the most general simply connected plane region," by W. F. Osgood; "'D' lines on quadrics," by A. Pell; "Sundry metric theorems concerning n lines in a plane," by F. H. Loud; "An application of group theory to hydrodynamics," by E. J. Wilczynski; "Determination of an abstract simple group of order $2^7.3^6.5.7$ holoedrically isomorphic with a certain orthogonal group and with a certain hyperabelian group," by L. E. Dickson.

In the October (closing) number of volume I. of the *Transactions* it is proposed, for the convenience of the readers, to collect under an appropriate heading brief notes concerning the memoirs of the volume. For this purpose the contributors are requested to furnish a complete list of errata in their memoirs as published, and any further notes, for instance in the way of reference.

The German mathematical society held its annual meetat Aachen, September 16th–23d, 1900, with Professor D. Hilbert as president, and Professor A. Gutzmer as secretary. The preliminary programme of papers to be presented is as follows:

(1). W. Ebert, Paris: "On the direct measurement of the motion of the earth's axis of rotation by means of stars near the pole."

(2) E. Jürgens, Aachen: "Computation of determinants."

(3) F. Klein, Göttingen: "Mechanics in the Encyclopædia of the mathematical sciences."

(4) A. Kneser, Dorpat: "On the development and present state of the calculus of variations."

(5) E. Kötter, Aachen: Subject to be announced.

(6) A. Marcuse, Berlin: "New development of the theory and practice of the determination of geographical and nautical positions."

(7) A. Marcuse, Berlin: "A new photographic universal instrument for the determination of geographic-astronomical positions."

(8) F. Meyer, Koenigsberg: "On singular binary forms and relations between sub-determinants."

(9) F. Meyer, Koenigsberg: "On geometrical theorems of the nature of those of Pascal and Desargues."

(10) H. Minkowski, Zürich: "On the concepts: length, surface, and volume."

(11) P. H. Schoute, Groningen: "On the number of points, right lines, planes, etc., of space of n dimensions."

(12) H. Schubert, Hamburg: "Zur Schlick'schen Ausgleichung der Schiffsvibrationen."

(13) D. SINTZOW, Ekaterinoslaw: "On connexes in space."
(14) P. STÄCKEL, Kiel: "On the theory of geodetic lines."
(15) A. WANGERIN, Halle: "Proof of a theorem relative to lines of curvature."
(16) A. WANGERIN, Halle: "Determination of all surfaces having constant curvature."

GAUTHIER-VILLARS announces the recent publication of Leçons sur la théorie des formes et la géométrie analytique supérieure, volume I, by H. ANDOYER; Recueil de problèmes de géométrie analytique, by FRANCOIS MICHEL; Analyse infinitésimale a l'usage des ingénieurs, volume I, by E. ROUCHÉ and L. LÉVY. The same publishing house has in preparation a translation into French of the Encyclopædia der mathematischen Wissenschaften, the first three volumes of which are in the editorial charge of JULES MOLK. The first volume to appear in print is volume II of the work which is expected from the press early in 1901. Among other works in preparation are a Cours d'analyse mathématique, by E. GOURSAT and an Essai sur les fondements de géométrie by W. B. RUSSELL. The second volumes of the above-named works of ANDOYER and of ROUCHÉ and LÉVY are in press.

THE Macmillan Company have in preparation an Introduction to celestial mechanics by Dr. F. R. MOULTON, associate professor of astronomy in the University of Chicago, and The principles of mechanics, by Professor FREDERICK SLATE, of the University of California.

THE several universities below offer during the winter semester 1900–1901 courses in mathematics as follows:

UNIVERSITY OF BERLIN.—By Professor L. FUCHS: Theory of elliptic functions, four hours; Introduction to the theory of differential equations, four hours; Seminar, two hours.—By Professor H. A. SCHWARZ: Differential calculus, four hours; Applications of the theory of elliptic functions, two hours; Calculus of variations, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of algebraic equations, four hours; Seminar, two hours.—By Professor J. KNOBLAUCH: Analytical geometry, four hours; Definite integrals, four hours; Exercises, one hour.—By Professor K. HENSEL; Integral calculus, four hours; Theory of numbers, four hours; Colloquium, two hours.—By Professor R. LEHMANN-FILHÉS: Analytical mechanics, four hours; Exercises, one hour.

UNIVERSITY OF BONN.—By Professor L. HEFFTER: Analytical geometry, four hours; Elliptic functions, four hours; Exercises in geometry, one hour.—By Professor H KORTUM: Infinitesimal calculus II, four hours; Curved lines and surfaces, two hours; Mathematical seminar, two hours.—By Professor R. LIPSCHITZ: Analytical mechanics, four hours; Mathematical seminar, two hours.

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Algebraic equations, four hours; Elements of the theory of differential equations, two hours; Seminar, one hour.—By Professor R. STURM: Infinitesimal calculus, four hours; Theory of numbers, two hours; Seminar, one hour.—By Dr. F. LONDON: Analytical mechanics, four hours, with exercises, one hour.

UNIVERSITY OF ERLANGEN.—By Professor M. NOETHER: Infinitesimal calculus, four hours; Introduction to algebraic analysis and the theory of functions, four hours; Mathematical exercises.—By Professor P. GORDAN: Analytical Geometry, four hours; Algebra, four hours; Seminar.

UNIVERSITY OF FREIBURG.—By Professor J. LÜROTH: Analytical mechanics, five hours; Trigonometry, three hours; Seminar, one hour.—By Professor L. STICKELBERGER: Plane analytical geometry, four hours; Theory of numbers, three hours.—By Dr. A. LOEWY: Differential calculus, three hours; Theory of surfaces, three hours; By Dr. E. REBMANN: Methodology of mathematical instruction, two hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Infinitesimal calculus, four hours; Selected chapters of analytical geometry, two hours; Exercises in the elements of higher mathematics, one hour; Seminar, one hour.—By Professor E. NETTO: Analytical geometry of space, four hours; Algebraic equations, two hours; Seminar, one hour.—By Professor R. HAUSSNER: Introduction to the theory of differential equations, four hours; Technical mechanics, four hours; Theory of determinants, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Projective geometry, four hours, with exercises, two hours.

—By Professor D. HILBERT: Theory of functions, four hours; Partial differential equations, four hours, with seminar in the latter, two hours.—By Professor M. BRENDEL: Rotational motion of the heavenly bodies, two hours; Exercises in numerical reckoning, two hours.—By Professor E. WIECHERT: Analytical mechanics, four hours.—By Professor F. SCHILLING: Integral calculus, four hours, with exercises, one hour.—By Professor AMBRONN: Method of least squares, two hours.—By Professor K. BOHLMANN: Mathematical principles of insurance, three hours; Mathematical statistics, two hours; Seminar in insurance, two hours.—By Dr. E. ZERMELO: Mengenlehre, two hours; Kinetic theory of gases, two hours.—By Dr. J. SOMMER: Theory of algebraic functions, two hours.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Mechanics I, four hours; Algebra, four hours; Seminar, two hours.—By Professor E. STUDY: Infinitesimal calculus II, four hours; Theory of functions II, four hours; Seminar, two hours.

UNIVERSITY OF HALLÈ.—By Professor G. CANTOR: Infinitesimal calculus, five hours.—By Professor A. WANGERIN: Chapters in higher geodesy, one hour; Integral calculus with exercises, four hours; Applications of elliptic functions, two hours; Calculus of variations, two hours.—By Professor V. EBERHARD: Theory of algebraic equations, four hours; Numerical equations with exercises, one hour.—By Dr. H. GRASSMANN: Ordinary differential equations, two hours; Exercises in descriptive geometry, one hour.—By Dr. E. NEUMANN: Theory of continued fractions, two hours.

UNIVERSITY OF HEIDELBERG.—By Professor L. KOENIGSBERGER: Higher algebra, four hours; Elements of the theory of differential equations, two hours; Theory of numbers, two hours; Seminar, two hours.—By Professor H. VALENTINER; Method of least squares, two hours.—By Professor M. CANTOR: Infinitesimal calculus, four hours, with exercises, one hour; Political arithmetic, two hours.—By Professor F. EISENLOHR: Theoretical optics, four hours; Infinitesimal calculus, five hours; Potential, two hours.—By Professor K. KOEHLER: Synthetic geometry of space, three hours.—By Professor G. LANDSBERG: Descriptive geometry, four hours; Selected chapters of theory of functions, continued from preceding semester, two hours.—By Dr. K. BOEHM: The principles of Heinrich Hertz's mechanics, two hours; Gauss's memoir on the theory of surfaces, one hour.

UNIVERSITY OF INNSBRUCK.—By Professor O. STOLZ: Real differential and integral calculus, four hours; Theory of functions of a complex variable after Cauchy and Weierstrass, three hours.—By Professor W. WIRTINGER: Linear partial differential equations, three hours; Eulerian integrals, two hours; Mathematical seminar, two hours.—By Dr. K. ZINDLER: Descriptive geometry with exercises, four hours; Application of the differential calculus to geometry, two hours.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Application of infinitesimal calculus to geometry, four hours; Definite integrals and hypergeometric series, two hours; Seminar, two hours.—By Professor A. GUTZMER: Integral calculus, four hours, with seminar exercises, one hour; Elements of descriptive geometry with exercises, four hours. —By Professor G. FREGE: Analytical geometry of space, four hours; Concepts, one hour.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Application of infinitesimal calculus to geometry, three hours; Introduction to the theory of functions, three hours; Seminar, two hours.—By Professor P. STÄCKEL: Higher analysis II, four hours; Stereometric drawing, four hours; Calculus of variations, one hour; Seminar, one hour.

UNIVERSITY OF KOENIGSBERG.—By Professor F. MEYER: Selected chapters of higher geometry, two hours; Integral calculus, three hours, with exercises, one hour; Seminar, one hour and a half.—By Professor A. SCHOENFLIES: Mechanics, four hours.—By Professor L. SAALSCHÜTZ: Theory of calculus of finite differences, two hours; Introduction to algebraic analysis, three hours, with exercises, one hour.— By Dr. E. MÜLLER: Analytical geometry of space, two hours; Perspective, two hours.

UNIVERSITY OF LEIPSIC.—Professor SCHEIBNER offers no courses this semester, and those of Professor MAYER will be announced later.—By Professor C. NEUMANN: Application of the infinitesimal calculus to geometry, four hours; Seminar, one hour.—By Professor O. HÖLDER: Infinitesimal calculus, five hours; Application of elliptic functions, one hour; Mathematical seminar, one hour.—By Professor L. BOLTZMANN: Analytical mechanics, five hours.—By Professor F. ENGEL: Projective geometry and homogeneous coordinates, three hours; Differential invariants, two hours; Seminar, one hour.—By Dr. F. HAUSDORFF: Theory of probability, three hours; Map projection, two hours.—By Dr. G. KOWALWOSKI: Theory and application of determinants, two hours; Theory of partial differential equations, two hours; Seminar in differential invariants, in conjunction with Professor Engel, one hour.—By Dr. H. LIEBMANN: Elementary theory of numbers, two hours; Foundations of geometry, two hours.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY: Elliptic functions, four hours; Algebraic analysis, three hours; Seminar, two hours.—By Professor E. HESS: Integral calculus, five hours; Selected chapters of geometry, three hours; Spherical trigonometry, two hours; Seminar, two hours.—By Dr. F. v. DALGWIK: Analytical geometry of the plane and space, five hours, with exercises, one hour; Descriptive geometry, with exercises, two hours.

UNIVERSITY OF MUNICH.—By Professor G. BAUER: Algebra; subject of second course to be announced; Seminar.—By Professor F. LINDEMANN: Plane analytical geometry; Introduction to ordinary and partial differential equations; Mathematical principles of life insurance; Seminar.—By Professor A. PRINGSHEIM: Differential calculus, with exercises; Elliptic functions.—By Dr. K. DÖHLEMANN: Descriptive geometry, with exercises; Kinematics.—By Dr. E. v. WEBER: Introduction to analysis; Determinants with applications; Selected chapters of elementary geometry.—By Dr. J. GÖTTLER: Theory and application of the functions of Lamé, Bessel and Laplace.

THE STEINER PRIZE.—The Academy of sciences of Berlin at the Leibniz Meeting, July 4, 1895, announced the following problem for the Steiner Prize: "To completely solve any important, hitherto unsolved problem relating to the theory of curved surfaces, taking into account, so far as possible, the methods and principles evolved by Steiner. It is required that sufficient analytical explanations shall accompany the geometrical investigations to verify the correctness and completeness of the solution. Without wishing to limit the choice of subject, the Academy takes the opportunity to call attention to the special problems to which Steiner has referred in his general remark at the end of his second paper on maximum and minimum in figures in a plane, on a sphere, and in space."

No paper on this subject was received. In accordance with the terms of the Steiner foundation, the Academy has utilized the prize of 6,000 marks, thus unawarded, for the purpose of recognizing certain important geometrical contributions published during the last few years. One third of it was awarded to each of the following: To DR. KARL FRIEDRICH GEISER, professor at the polytechnic school at Zurich, for his individual researches in geometry and his services in the publication of a part of Steiner's lectures; to DR. DAVID HILBERT, professor at the University of Göttingen, for his important researches on the axioms of geometry and for the advancement which analytic geometry has experienced from his work on the theory of invariants; to DR. FERDINAND LINDEMANN, professor at the University of Munich, who has earned special distinction in geometry by his celebrated discussion of the quadrature of the circle, as well as by editing Clebsch's Vorlesungen über Geometrie.

For the year 1905 the Academy again announces the foregoing problem, which has remained unsolved. For its solution a prize of 4,000 marks is offered, with an additional sum of 2,000 marks. Papers offered in competition may be written in German, French, English, Italian or Latin, and must be submitted before December 31, 1904. The result will be announced at the Leibniz meeting of 1905. No manuscript revealing the name of the author will be accepted. Each manuscript is to bear a mark or nom-de-plume, and to be accompanied by a sealed envelope containing the name and address of the author and bearing outside the corresponding mark or assumed name. Manuscripts should be sent to the Bureau of the Academy, Berlin NW. 7, Universitäts-strasse, 8.

FROM an article in *Science*, August 31, 1900, it appears that the degree of doctor of philosophy was conferred by American universities during the past year on 233 candidates, of whom 11 presented dissertations in mathematics. The list of mathematical doctorates, with titles of theses and names of universities conferring the degrees, is as follows: L. P. EISENHART, Johns Hopkins University, "Infinitesimal deformation of surface"; W. GILLESPIE, University of Chicago, "Determination of all hyperelliptic integrals of the first kind of genus 3 reducible to elliptic integrals by transformations of the second and third degrees"; H. E. HAWKES, Yale University, "Examination and extension of Peirce's linear associative algebra"; D. N. LEHMER, University of Chicago, "Asymptotic evaluation of certain totient sums"; J. H. MCDONALD, University of Chicago, "Concerning the system of the binary cubic and quadratic with application to the reduction of hyperelliptic integrals to elliptic integrals by a transformation of order four"; C. R. McInnes, Johns Hopkins University, "Superosculated sections of surfaces"; H. C. Moreno, Clark University, "On ruled loci in n-fold space"; F. R. Moulton, University of Chicago, "A particular class of periodic solutions of the problem of three bodies"; S. E. Slocum, Clark University, "On the continuity of groups generated by infinitesimal transformations"; F. B. Williams, Clark University, "Geometry on ruled quartic surfaces"; A. D. Yocum, University of Pennsylvania. "An inquiry into the fundamental processes of addition and subtraction."

The Hopkins prize of Cambridge University for the period 1894–1897 has been awarded to Mr. J. Larmor of St. John's College for his investigation on the "Physics of the æther" and other contributions to mathematical science.

The works of the late Professor E. Beltrami are to be published by the faculty of science of the University of Rome. The edition, which is to be issued by subscription, will consist of three or four large volumes. A complete list of Professor Beltrami's published papers appeared in the *Annali di Matematica* for March, 1900.

The recent catalogue, No. 127, of M. Hoepli, Milan, Italy, contains nearly three thousand titles of memoirs and treatises in mathematics and physics.

Professor R. Lipschitz, of Bonn, has been elected a correspondent of the section of geometry and Professor P. Duhem correspondent of the section of mechanics of the Paris academy of sciences.

Professor F. Klein, of Göttingen, has been elected corresponding member of the Vienna academy of sciences. At the recent celebration of its five-hundredth anniversary, the University of Cracow conferred on Professor Klein an honorary doctorate.

Dr. J. H. Gore, professor of mathematics and geodesy at the Columbian University, has been appointed juror-in-chief of the Court of Appeal of the Paris Exposition.

Professor A. R. Forsyth has been elected an honorary member of the Edinburgh Royal Society.

Dr. Arnold Emch, recently professor of graphic mathematics in the Kansas State Agricultural College, has been appointed to an assistant professorship of mathematics in the University of Colorado.

Dr. Louis T. More, of the University of Nebraska, has been elected professor of physics in the University of Cincinnati.

Dr. Jacob Westlund, of Yale University, has been appointed instructor in mathematics in Purdue University.

Dr. H. Fehr, docent at the University of Geneva, has been promoted to a full professorship in geometry and algebra at the same university.

Dr. Karl Boehm has been made docent in mathematics, at the University of Heidelberg.

Mr. W. B. Fite and Mr. H. B. Kuhn, recent holders of fellowships, have been appointed assistants in the department of mathematics at Cornell University.

Professor C. S. Venable, emeritus professor of mathematics in the University of Virginia, died August 11, 1900, at his home in Charlottesville, Va., at the age of 73 years. His education was received at Hampden Sidney College and the University of Virginia, and at the Universities of Berlin and Bonn. He served in the Confederate army with distinction throughout the civil war, most of the time as a member of Gen. Lee's staff. Before the war he was successively a member of the faculties of Hampden Sidney College, the University of Georgia, and the College of South Carolina; and since the war he served continuously as professor of mathematics in the University of Virginia until in 1896 failing health induced him to resign.

As the author of a series of well known text books on mathematics and by his work at the University of Virginia, he exerted a profound influence upon mathematical instruction throughout the Southern States. As a pupil of Encke and Argelander, he became deeply interested in astronomy. In 1860 he observed the total eclipse of the sun in Labrador, and more recently it was mainly through his exertions that the establishment and endowment of the Leander McCormick Observatory was secured.

Professor J. E. Keeler, director of the Lick Observatory, died at San Francisco, August 12, 1900, aged 43 years.

. THE death is announced of Dr. G. BRUNEL, professor of mechanics and dean of the faculty of sciences of Bordeaux.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDOYER (H.). Leçons sur la théorie des formes et la géométrie analytique supérieure, à l'usage des étudiants des facultés des sciences. (En 2 volumes.) Vol. I. Paris, Gauthier-Villars, 1900. 8vo. 6 + 508 pp. Fr. 15.00

ARCAIS (F. D'). Corso del calcolo infinitesimale. Vol. II, parte 1 2a edizione, con aggiunte e modificazioni. Padova, Draghi, 1900. 8vo. 368 pp. Prezzo dell'opera completa in due volumi Fr. 18.00

BACHMANN (P.). See ENCYKLOPÄDIE.

BALL (R. S.). A treatise on the theory of screws. London and New York, Macmillan, 1900. 8vo. 19 + 544 pp. (Cambridge University Press.) Cloth. $5.50

BÔCHER (M.). See ENCYKLOPÄDIE.

BOUTROUX (P.). L'imagination et les mathématiques, selon Descartes. Paris, Alcan, 1900. 8vo 51 pp. (Bibliothèque de la Faculté des lettres de l'Université de Paris, No. X.) Fr. 2.00

BURKHARDT (H.) See ENCYKLOPÄDIE.

BURONI (S.). See LAVAGGI.

CAHEN (E.). Eléments de la théorie des nombres : Congruences ; formes quadratiques ; nombres incommensurables ; questions diverses. Paris, Gauthier-Villars, 1900. 8vo. 8 + 403 pp. Fr. 12.00

CALINON (A.). Etude de géométrie numérique. Paris, Gauthier-Villars, 1900. 8vo. 35 pp. Fr. 2.00

COMBEROUSSE (C. DE). Cours de mathématiques, à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, à l'Ecole centrale des arts et manufactures. Vol. I : Arithmétique et algèbre élémentaire. 4e édition, revue et augmentée. Paris, Gauthier-Villars, 1900 8vo. Fr. 10.00

COMPLETE INDEX of all the papers printed in the *Proceedings of the London Mathematical Society*, vols. 1-30. London, Hodgson, 1900. 12mo. 32 pp.

COX (H.). Rudimentary treatise on integral calculus. London, 1900. 8vo. 136 pp. Cloth. 1s. 6d

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THE INTERNATIONAL CONGRESS OF MATHEMATICIANS IN PARIS.

At the Zürich Congress of 1897 it was agreed to hold the next congress in Paris in 1900, the French Mathematical Society being charged with the preparations. Circulars have been issued at intervals during the last eighteen months, calling the attention of mathematicians to the arrangements in progress. The congress was finally announced for August 6th–11th, and the opening general meeting was held in the Palais des Congrès, in the Exhibition grounds, at 9.30 on the morning of Monday, August 6th. M. Poincaré was elected President, M. Hermite, who of course was not present, being the *Président d'honneur*. The executive board was constituted as follows: vice-presidents, MM. Czuber (Vienna), Geiser (Zürich), Gordan (Erlangen), Greenhill (London), Lindelöf (Helsingfors), Lindemann (Munich), Mittag-Leffler (Stockholm), Moore (Chicago, absent), Tikhomandritzky (Kharkoff), Volterra (Turin), Zeuthen (Copenhagen); secretaries, MM. Bendixson (Stockholm), Capelli (Naples), Minkowski (Zürich), Ptaszycki (St. Petersburg), Whitehead (Cambridge, absent); general secretary, M. Duporcq (Paris). After the announcement of the officers of the sections and the names of the official delegates, and a very few words from the President, the two addresses of the day, both in French, were delivered by MM. M. Cantor (Heidelberg) and Volterra (Turin); each occupied about three-quarters of an hour.

M. Cantor: *Sur l'historiographie des mathématiques.*

During the century drawing to its close the character of mathematics has changed; its devotees are now differentiated into geometers, analysts, algebraists, arithmeticians, astronomers, theoretical physicists, and historiographers. These last make no claim to advancing the science itself; they press neither towards the arctic pole of the theory of functions, nor towards the antarctic pole of algebra; they explore neither the steep surfaces of geometry nor the depths of differential equations. Their task is rather to draw up guides and charts, to indicate by what routes the results have been obtained, and what important points have been passed by without sufficient exploration. This work began with the History of Eudemus of Rhodes, B. C.

300, of which only a fragment has been preserved, just sufficient to excite lively regret for the loss of the whole. During the next two thousand years there were many bald chronicles of mathematics, but historiography as a science begins with Montucla. Notwithstanding the errors, unavoidable at that time, to be found in the two volumes of his Histoire des mathématiques (1st edition, 1758, 2d edition with two volumes by Lalande, 1799), Montucla "est encore et restera peut-être toujours un modèle que tout historiographe des sciences doit suivre." Kästner published four volumes of his Geschichte der Mathematik in the last four years of his life, 1796–1800. He has been alternately over-praised and depreciated; Gauss referred to him as the best poet among the mathematicians, the best mathematician among the poets of his day. His history is no real history, it is rather a catalogue *raisonné*, but it is nevertheless invaluable, on account of its conscientious analysis of a number of works, which, with their authors, would be otherwise absolutely unknown to us now. At about the same date, 1797–1799, there appeared the two volumes of Cossali's Storia critica dell'algebra, dealing exhaustively with the period 1200–1600; as regards Italy only, it is true, but then during this period the Italian algebra was of importance far surpassing that of any other country. Cossali's labors for the elucidation of Leonard of Pisa and Cardan are of special merit.

Bossut published in 1810 his Histoire des mathématiques; in this he gives only rapid *aperçus* of the general development, interesting to those that know already, useless to those that need to learn. In the present century we have first Chasles, to whom the speaker paid a warm personal tribute. In his Aperçu historique published in 1837, the notes, dealing with geometry, calculation, algebra, mechanics, which attain the dimensions of memoirs, form the model part of the volume, the text, the actual "Aperçu," being but a very condensed statement of the history of synthetic geometry. The other historical work of Chasles, the Rapport sur les progrès de la géométrie of 1870, is seriously affected by his ignorance of the German language. The years 1837–1841 saw the publication of Libri's Histoire des sciences mathématiques en Italie, from the earliest times up to the middle of the 17th century, a work which owing to the author's admirable style "se lit comme un roman, même dans les parties où elle n'en est pas un." Notwithstanding Libri's immense services in the study of manuscripts, his history is vitiated, as a historical work, by his misplaced patriotism; according to him all progress in mathematics is due to the Italians, with perhaps a few scattered French writers. When he finds an Italian in possession of any ideas or methods, no matter whence derived, he at once credits him with their discovery. In any case, it is not possible to give any true idea of the history of mathematics by tracing it in one country only. If there is an international science, it is mathematics; it bears no stamp of nationality. In considering the earliest times, it is impossible to understand the course of mathematics in one country without following it in others also; to understand Greek mathematics, we must know something of Egypt and Babylonia; the mathematics of the Arabs cannot be explained without reference to Egypt, Greece, and India. After the invention of printing, so long as Latin was in use, mathematics had no country; and even when the frontiers were faintly marked by the use of different languages, they were speedily obliterated for most mathematicians.

Passing rapidly over Gerhardt and Quetelet, with a few words of recognition, M. Cantor spoke of Nesselmann's Die Algebra der Griechen, 1842, "un chef d'œuvre digne d'être mis a côté de l'Aperçu historique de Chasles"; of Arneth's Geschichte der reinen Mathematik, 1852, which would have been an excellent book, if the author had made a better apportionment of his space to his material—parts of the work "fourmillent de remarques aussi spirituelles que profondes"; of Hankel's posthumous fragment, 1876, "un torse d'une telle beauté qu'il eut été pitié de ne pas le mettre au grand jour"; and of the prince Baldassare Boncompagni's disinterested labors on behalf of historiography. In this sketch he passed over many authors "tous aussi morts que leurs livres; gardons-nous de les ressusciter"; and avoided all mention of living authors for very obvious reasons. He brought his address to a close by a forecast of the mode in which the history of more recent mathematics must be written. Regarding Lagrange as the founder of modern mathematics. this gives 1759 as the starting point; and from this year on, the different subjects will have to be treated in special volumes. This however will be insufficient; the development of the lines of thought that run through all these different branches of mathematics must be traced in one final volume, the History of Ideas; difficult to write, certainly, but indispensable, for as Jacobi said, "Mathematics is a science of which it is impossible to understand any one part without knowing all the others."

V. VOLTERRA: *Trois analystes italiens; Betti, Brioschi, Casorati.*

The scientific existence of Italy as a nation dates from a journey which Betti, Brioschi, and Casorati took together in the autumn of 1858, with the object of entering into relations with the foremost mathematicians of France and Germany. It is to the teaching, labors, and devotion of these three, to their influence in the organization of advanced studies, to the friendly scientific relations that they instituted between Italy and foreign countries, that the existence of a school of analysts in Italy is due.

The extent of their joint influence, affecting minds of many diverse casts, is largely due to the differences in their natural faculties, in the circumstances of their lives, and in their acquired tendencies. Brioschi, "toujours jeune par son caractère et toujours mûr par son esprit," a Lombard by birth, was at first an engineer; but at an early age he acquired a profound knowledge of the classical mathematical works, and was called to the chair of mechanics at Pavia at the age of 25. He founded the Polytechnic School at Milan, and held the directorship until his death; in his capacity of Senator, he was active in public affairs; he found time to engage in public works and in engineering; and up to the last, as Director of the *Annali di Matematica*, and President of the Accademia dei Lincei, he was one of the leaders of the mathematical movement in Italy. A great contrast to this active life is offered by the calm existence of Betti. He was born in a mountain village in Tuscany; at 34 he became a professor in the University of Pisa, and at 41 Director of the Scuola normale superiore of Pisa, whose organization is much like that of the École normale supérieure of Paris; he took no part in political movements. He loved scientific researches for their own sake exclusively, without regard to the results they might produce in the scientific world, or to their importance in teaching. He did not care for publishing his researches; and even when he did undertake this, he was apt to push it aside, attracted by new ideas. The knowledge that his intellectual conception could be realized was all-sufficient for him; he did not give himself the trouble of carrying it out in detail. When once he had obtained a clear vision of hidden truths, and had constructed in his own mind a system in which they proceeded directly from the simplest principles, "tout était fait pour Betti."

Casorati was born and lived at Pavia; he passed through the various grades in the University, where at the time of his death he was professor of infinitesimal analysis. He lived and worked almost exclusively for his pupils; all his works bear the stamp of the practical teacher, bent on elucidating some obscurity, correcting some error, expounding some theory. All his writings were in a definite relation to his university teaching; in his mind there was no distinction between the work of the savant and the work of the professor.

The fundamental differences in the three can be brought out most clearly by a comparison of their attitude towards the theory of functions. The development of this theory exhibits three well-marked periods, corresponding to the three distinct phases that can be recognized in the history of any mathematical subject; these three phases, however, correspond also to three distinct modes of regarding questions in analysis, each of which has its advocates. In the first instance, the discovery of facts is all-important, and particular theories are elaborated. There are no uniform methods; every question is attacked on its own merits. and methods are created as occasion arises; the ideas and results disengage themselves finally from long calculations. In the theory of functions this manifests itself in the heroic period, personified in Euler, Jacobi, Abel; and this manner of approaching questions is natural to Brioschi, the engineer and practical man, with his extraordinary gift for dealing with formidable calculations. He remained faithful to the classical method, never attracted by the second phase, which he even scorned somewhat. In this second phase, ideas replace calculations; the philosophic spirit takes control and demands a general method including the whole subject in one body of doctrine. This desire found its fulfilment in the second period of the theory of functions. in the works of Cauchy, Weierstrass, and Riemann, who derive everything from the very sources of the fundamental conceptions. To this period belongs Betti the philosopher. His broad and cultivated mind loved philosophic systems; his Tuscan indolence (which is not intellectual idleness) caused him to delight in meditation rather than in mechanical labor. Curiously enough, his name is associated with the theory of Weierstrass just as surely as with that of Riemann; his education had made him an algebraist while nature meant him for a physicist.

In the final period the theories find their appropriate applications, their most suitable forms; they are refined by criticism, and cast into a didactic mould. The name of Casorati, critic and teacher, is associated with this third phase. His work, Teorica delle funzioni di variabili complesse, has served more than any other one book to popularize in Italy the fundamental conceptions of the theory of functions, for the reason that, while reading it, difficulties disappear. The influence of this book is not confined to professed analysts; anyone attempting to trace the development of mathematics in Italy during this half century will find that analysts and pure geometers have influenced one another. For instance, the ideas of Riemann are at the foundation of many of the works of Italian geometers, and while the actual introduction of these ideas was due to Betti, it is this book of Casorati's that has carried them everywhere and attracted the attention of geometers.

This comparison of the work of these three analysts in the region that they had in common gives no idea, however, of the extent of the labors and influence of each one. For this it would be necessary to dwell on the work of Casorati in the theory of differential equations, in analytical and infinitesimal geometry; of Betti in mathematical physics and algebra, he being one of the first to accept the new ideas of Galois; of Brioschi in mechanics, algebra, and geometry. The field in which Betti and Brioschi first obtained renown was in fact that of algebra; their names will always be associated with that of Kronecker as second only to Hermite in their work on the equation of the fifth degree, an equation whose complete solution was due to and secured immortality for M. Hermite.

This concluded the business of the first general meeting, with the exception of one or two formal announcements relating to secretarial matters. This was the only one of the meetings to be held in the Exhibition grounds; all the others were held at the Sorbonne. Six sections were organized for the presentation of special papers, to meet on the 7th, 8th, 9th, and 10th of August, as follows:

Section I, Arithmetic and Algebra; Tuesday, Thursday and Friday mornings; president, M. Hilbert, secretary, M. Cartan.

Section II, Analysis; Tuesday and Thursday mornings; president, M. Painlevé, secretary, M. Hadamard.

Section III, Geometry; Tuesday and Thursday afternoons; president, M. Darboux, secretary, M. Niewenglowski.

Section IV, Mechanics and Mathematical Physics; Tuesday and Thursday afternoons; president, M. Larmor, secretary, M. Levi-Cività.

Section V, Bibliography and History; Wednesday morning and afternoon and Friday morning; president, Prince Roland Bonaparte, secretary, M. d'Ocagne.

Section VI, Teaching and Methods; Wednesday morning and afternoon and Friday morning; president, M. Cantor, secretary, M. Laisant.

Sections V and VI, however, amalgamated and sat as one section, thus making the sections the same as those at Zürich. It hardly seems advisable to give a complete list of the papers, as all will be given in the full official report, which will appear shortly; it seems better to give some account of the most interesting.

In Section I the most noticeable communication was that of M. Henri Padé, of Lille, "Aperçu sur les développements récents de la théorie des fractions continues." The object of this communication was the discussion of the question as to what is to be understood by the development of a function as a continued fraction, and the examination of the consequences of the answer obtained. For the function e^x, for instance, five such developments are already known, due to Euler, Lagrange, and Gauss. Why five? By what are the five characterized? How are they related to one another? The bond that unites them consists in a certain property of approximation common to their convergents, and thus there arises the more general question of the study of rational fractions satisfying this condition of approximation for a given function.* This investigation yields the following results. Every function that is developable by Maclaurin's formula gives rise to an infinity of developments as a continued fraction; as to form, these fractions present the common characteristics that all the partial numerators are monomials, with coefficient and exponent different from zero; as to matter, they are characterized by the property that all the convergents satisfy the condition of approximation mentioned above, and give an approximation whose order increases constantly as we pass from any convergent to the following one.

Among these fractions, called *fractions continues holoïdes* of the function, are included the regular continued fractions, and it is among these last that are found, as a very particular case, the developments in continued fractions already known for some special functions, for example, the five developments of e^x.

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* "Sur la représentation approchée d'une fonction par des fractions rationnelles," *Ann. scient. de l' Ecole norm. sup.*, vol. 9 (1892).

M. Padé then indicated the two ways in which these results can be generalized, the extension they involve in all the applications of continued fractions hitherto made, and the important consequences to which they lead, both in the theory of functions, where they have already introduced the question of the use of divergent power-series, and in the theory of numbers.

In Section II, the first paper read on Tuesday morning was M. Tikhomandritzky's "L'évanouissement des fonctions θ de plusieurs variables indépendantes." The function $\theta\Big(u_k \overset{\xi p}{\underset{1}{-}} \underset{x_0}{I}_k\Big)$ of p independent variables $u_k = \sum_{\iota=1}^{p} \overset{x^\iota}{\underset{a_\iota}{I}}_k$ vanishes 1° when some of the points $\Big(\overset{p}{\underset{1}{x_\iota, y_\iota}}\Big)$ fall at (ξ, y_ξ); 2° when they are on an adjoint curve of the first kind, $\varphi(x^{m-2}, y^{n-2}) = 0$. If with Weierstrass we define this function by the equation

$$\theta\Big(u_k \overset{p}{\underset{1}{-}} \overset{\xi}{\underset{x_0}{I}}_k\Big) = e^{\int \Sigma\left[\sigma_\lambda + J\left(u_k \overset{p}{\underset{1}{-}} I_k\right)_\lambda\right] du_\lambda}, \tag{1}$$

where

$$J\Big(\overset{p}{\underset{1}{u_k}}\Big)_\lambda = \sum_{\iota=1}^{p} \overset{x_1}{\underset{x_0}{\Pi}}_k \tag{2}$$

($\overset{x_\iota}{\underset{x_0}{\Pi}}_k$ denoting the integral of the second species, which becomes infinite when (x_ι, y_ι) falls at $(a_k, b_\lambda,)$), this property of θ must be derived from those of the function (2). For this purpose, considering in the first place the function

$$J\Big(u_k \overset{p}{\underset{1}{+}} \overset{x'}{\underset{\xi}{I}}_k\Big)_\lambda = \sum_{\iota=1}^{p} \overset{a_\iota}{\underset{x_0}{\Pi}}_\lambda \tag{3}$$

((x', y') denoting a point very near to (a_λ, b_λ)), where the points (x', y'), $(\overset{p}{\underset{1}{x_\iota, y_\iota}})$ are the infinities, and (ξ, y_ξ), $(\overset{p}{\underset{1}{a_\iota, y_{a_\iota}}})$ the zeros, of the principal function of (z, y_z)

$$P_{x\xi}\Big(x', y'; \overset{p}{\underset{1}{x_\iota, y_\iota}}\Big) = \frac{P_{x', x_p}\Big(z, y;\ \overset{p-1}{\underset{1}{x_\iota', y_\iota'}};\ \xi, y_\xi \,\Big|\, \overset{p}{\underset{1}{a_\iota, y_{a_\iota}}}\Big)}{\varphi\Big(\overset{m-2}{z}, \overset{n-2}{y_z};\ \overset{p-1}{\underset{1}{x_\iota, y_\iota \,|\, x_\iota', y_\iota'}}\Big)} \tag{4}$$

we see, by the second form of this function, that in the two cases one of the infinities of the function in the numerator being absorbed by one of its arbitrary zeros (1° (x_p, y_p) by (ξ, y_ξ) ; 2° (x_p, y_p) by (x'_{p-1}, y'_{p-1})), the other will be absorbed by one of its non-arbitrary zeros $\left(\overset{p}{\underset{1}{a}}, y_{\overset{p}{\underset{1}{a}}}\right)$. Hence in the two cases, at the limit, for $(x', y') = (a_k, b_k)$, one of the integrals in (3) will become infinite like $\frac{1}{x - a_k}$ for $x = a_k$; thus θ will vanish.

More general interest was taken in M. Mittag-Leffler's papers, which followed. "Sur fonction analytique et expression analytique." "Une application de la théorie des séries n-fois infinies." "Sur une extension de la série de Taylor." In these M. Mittag-Leffler reported on his recent researches* in the theory of functions. Let $f(a)$, $f'(a)$, $f''(a), \cdots$ determine an element $P(x|a) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(a) \cdot (x - a)^n$ of an analytic function $f(x)$. This analytic expression is valid only for its circular domain of convergence; for a more extended region $P(x|a)$ must be supplemented by certain of its continuations. Let a lie within a continuous region which does not overlap itself, and let the branch of $f(x)$ derived from $P(x | a)$ by continuation throughout K be one-valued and regular; is it possible to find a *single* analytic expression for this branch $fK(x)$, where K is given its maximum extension, the formula to be based only on the primary quantities $f(a)$, $f'(a)$, $\cdots$? M. Mittag-Leffler has shown that such an expression can be built up in a comparatively simple manner. In doing this he has employed a new geometrical notion, that of the *star* (étoile). If a ray revolves about a, proceeding in each position to the nearest singular point of $f(x)$, this point lying it may be at infinity, the collection of points on the totality of such rays forms for $f(x)$ the star A ; it is assumed that the lower limit of these rays is not zero. For A we have the theorem that

$$fA(x) = \sum_{m=0}^{\infty} G_m(x),$$

where $G_m(x)$ denotes a polynomial $\sum c_{n,m} f^n(a)(x - a)^n$, in which the coefficients $c_{n,m}$ are given initially and do not depend on a or on $f(a), f'(a), \cdots$.

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* "Sur la représentation analytique d'une branche uniforme d'une fonction monogène ;" *Acta Math.*, vols. 23, 24 (1899, 1900).

The expression

$$G_n(x \mid a)$$

$$= \sum_{\lambda_1=0}^{n^2} \sum_{\lambda_2=0}^{n^4} \cdots \sum_{\lambda_n=0}^{n^{2n}} \frac{1}{\lambda_1! \cdots \lambda_n!} f^{(\lambda_1+\cdots+\lambda_n)}(a) \left(\frac{x-a}{n}\right)^{\lambda_1+\cdots+\lambda_n}$$

leads to a limiting expression $\lim_{n=\infty} G_n(x \mid a)$ with the following properties: It is uniformly convergent for every region interior to the star A, but never uniformly convergent for a region containing a vertex of A. Within A it represents the branch $fA(x)$ of $f(x)$.

It is perfectly possible that $\lim_{n=\infty} G_n(x \mid a)$ may converge outside A; the star A is not a star of convergence for $\lim_{n=\infty} G_n(x \mid a)$. M. Mittag-Leffler has shown that it is possible to replace $\lim_{n=\infty} G_n(x \mid a)$ by another expression for which A is a star of convergence. $\lim_{n=\infty} G_n(x \mid a)$ was obtained from an n-fold series in x by making the maximum values of $\lambda_1, \lambda_2, \cdots, \lambda_n$ proceed simultaneously to the limit ∞. If the passage to the limit is performed in another way, viz., by taking $\sum_{\lambda_1=0}^{\lambda_1} \sum_{\lambda_2=0}^{\lambda_2} \cdots \sum_{\lambda_n=0}^{\lambda_n}$ in place of $\sum_{\lambda_1=0}^{n^2} \sum_{\lambda_2=0}^{n^4} \cdots \sum_{\lambda_n=0}^{n^{2n}}$ and then making $\lambda_n, \lambda_{n-1}, \cdots, \lambda_1$ tend *successively*, in the order named, to infinity, the expression

$$S_n(x \mid a) = \sum_{\lambda_1=0}^{\lambda_1} \sum_{\lambda_2=0}^{\lambda_2} \cdots \sum_{\lambda_n=0}^{\lambda_n} c_{\lambda_1 \cdots \lambda_n} f^{\lambda_1+\cdots+\lambda_n}(a) \left(\frac{x-a}{n}\right)^{\lambda_1+\cdots+\lambda_n}$$

(where the c's are given numerical constants of which

$$c_{\lambda_1} = \frac{1}{\lambda_1!}, \quad c_{\lambda_1\lambda_2} = \frac{1}{\lambda_1!\,\lambda_2!}\left(\frac{1}{2}\right)^{\lambda_1+\lambda_2}, \quad c_{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_1!\,\lambda_2!\,\lambda_3!}\left(\frac{1}{3}\right)^{\lambda_1+\lambda_2+\lambda_3},$$

while for values of $n > 3$ they are algebraic irrationals) yields the desired result; $\lim_{\infty} S_n(x \mid a)$ has the star A as a star of convergence, and represents $fA(x)$ within A. Writing $n = 1$, the series is seen to be simply Taylor's series; in general it is an extension of Taylor's series.

In the course of his remarks M. Mittag-Leffler referred to recent researches of M. Borel; this led to a discussion in which MM. Borel, Hadamard, and Painlevé took part, on the nature of the connection between "analytic expression in a complex variable x" and "analytic function in x."

In Section III papers were presented by MM. Lovett, "On contact transformations between the essential elements of space"; Macfarlane, "Applications of space analysis to curvilinear coördinates"; Stringham, "Orthogonal transformations in elliptic or in hyperbolic space"; Amodeo, and others. In Section IV very few papers were read; one appointed meeting of the section was not held, and some of the papers intended for the section were presented at the joint sitting of Sections V and VI, which was transformed momentarily into a sitting of Section IV, to hear MM. Hadamard: "Relations entre les caractéristiques réelles et les caractéristiques imaginaires pour les équations différentielles à plusieurs variables indépendantes"; and Volterra: "Comment on passe de l'équation de Poisson à caractéristique imaginaire à une équation semblable à caractéristique réelle."

The communications made in Sections V and VI, while not necessarily the most valuable mathematically, were yet of the most general interest, and lend themselves best to any general report. The sitting was opened by M. Hilbert's address, in German, on the future problems of mathematics. The lines along which we may expect the development of any science which is progressing in a continuous manner can be detected by an examination of the problems to which attention is specially paid. Among these most importance is to be attached to those that are sharply defined and stand out well; such, for example, as the problem of three bodies. In the earlier stages of any science, problems present themselves naturally through experience, as is exemplified in mathematics by the duplication of the cube and the quadrature of the circle, and at a later date by the questions arising with reference to infinitesimal analysis and the theory of the potential; but as the science progresses, it is the logical faculty of the intellect that imposes on us problems such as are found in the theories of prime numbers, elliptic functions, etc.

As to our aim with regard to any problem, there must be a definite result of some kind, it cannot be laid aside until we have obtained either a satisfactory solution or a rigorous demonstration of the impossibility of a solution. The mathematical rigor that is essential in the treatment of a problem does not require complicated demonstrations; it requires only that the result be obtained by a finite number of logical steps from a finite number of hypotheses furnished by the problem itself; in seeking this rigor we may find simplicity. The proper treatment of any problem depends on 1° a complete system of axioms, by means of which the conceptions are defined, 2° a system of symbols appropriate to the conceptions with which the problem deals; thus a demonstration by means of geometrical symbols is as legitimate as an arithmetical one, provided that the axioms on which it is based are perfectly understood. The mere formulation of these axioms is in some cases itself the problem, as for instance in arithmetic and physics. Among the ten problems that M. Hilbert specified in particular as fitted to advance mathematics, No. 2 is that of finding some one system of independent compatible axioms governing and defining arithmetical conceptions, and No. 3 is the same question for the calculus of probabilities, rational mechanics, and physics. Other problems are to prove that $e^{i\pi z}$ is transcendental when z is an algebraic irrational; and that the solution of the general equation of the 7th degree cannot be obtained by means of a finite number of operations involving only two parameters. In geometry, the relative situation of the circuits that a plane curve of assigned order can possess, with the corresponding question as regards surfaces, demands investigation; in the theory of functions there is the question of the expression of two variables, connected by any analytic relation whatever, as uniform functions of a single parameter z—for Poincaré's theorem (*Bulletin de la Société mathématique de France*, volume 11 (1883)) is subject to some limitations. These are but a few of the problems that M. Hilbert mentioned, and these were a selection from a much longer list for which he referred to an article about to appear in the *Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen*, 1900. In the course of a rather desultory discussion that followed the reading of this paper, the claim was made, though apparently without adequate grounds, that more had been done as regards the equation of the 7th degree (by some German writer) than the author of the paper was willing to allow. A more precise objection was taken to M. Hilbert's remarks on the axioms of arithmetic by M. Peano, who claimed that such a system as that specified as desirable has already been established by his compatriots MM. Burali-Forti, Padoa, Pieri, in memoirs referred to on pp. 3–5 of no. 1, volume 7 of the *Rivista di Matematica*.

M. Hilbert was followed by M. Fujisawa, the official delegate from Japan, who gave, in English, a very interesting account of the mathematics of the older Japanese school. It is difficult to follow the course of Japanese mathematics; there are some two thousand manuscript volumes still to be transcribed; in these much valuable work is mixed up with what is purely elementary and even trivial. The difficulty of arriving at any clear idea is greatly increased by the fact that publication of results was not customary; they were preserved to a great extent only by oral transmission. So far as the books have been deciphered and collated, one fact stands out with ever-increasing clearness, and that is that side by side with one less important school of Japanese mathematics there exists another earlier system of mathematics of a peculiar kind, which had its origin in Japan, and was developed there entirely free from any external influences.

The mathematics of the first kind, probably derived from the Chinese at a very early date, displays a noticeable lack of rigor; for instance, $\sqrt{10}$ is accepted as the value of π. As to content; in algebra, the solution of simple equations and the formulæ for the sum of an arithmetical or geometrical progression were known; in geometry, the right-angled triangle with sides proportional to 3, 4, 5 was used, with some propositions regarding regular polygons; magic squares were discussed, even so far as those containing the first 400 numbers. Bamboo rods were used for purposes of calculation; these were placed one above another to indicate addition, side by side to indicate multiplication, diagonally to denote subtraction.

The other part of Japanese mathematics, that indigenous to the country, is of more importance and interest. It appears that the mathematicians of this school made use of local value in expressing numbers, invented zero for themselves, and used the circle as the symbol for zero. They were familiar with imaginaries and complex numbers; and were such adepts at calculation that they found the value of π correct to 49 places of decimals. M. Fujisawa explained that the knowledge of this part of Japanese mathematics so far obtained is very fragmentary, the unexplored part offers an attractive field of research for Japanese who may care to devote themselves to it. It is a matter of purely historical interest, as the present teaching of mathematics in Japan is in no sense founded on it; for, very wisely as he thinks, the Japanese educational authorities made an entirely fresh start, sweeping away all trace of this older educational system.

The president of the section, M. Cantor, then spoke of the difficulties he encountered, when writing his Geschichte der Mathematik, in finding out anything about the earlier Japanese mathematics. When he did finally hear of a work of reference it turned out to be written in Japanese. With reference to the earliest use of zero, he expressed the opinion that it was probably due to the Babylonians, about 1700 B. C.

Another paper of interest in these sections was that of M. Padoa (Rome) on Friday morning: "Un nouveau système irréductible de postulats pour l'algèbre." Naming the object, *entier* (integer), two undefined derivatives, *successif* and *symétrique*, are considered. The seven postulates are

1°. If a is an integer, then suc. a is an integer.
2°. If a is an integer, then sym. a. is an integer.
3°. If a is an integer, then sym. (sym. a) $= a$.
4°. If a is an integer, then sym. $\{$suc. [sym. (suc. a)]$\} = a$.
5°. There exists an integer x such that sym. $x = x$.
6°. There do not exist two different integers x, y, such that sym. $x = x$, and sym. $y = y$.
7°. If a class u of objects satisfies the conditions
 (i) it contains some one integer,
 (ii) if it contains an integer x it contains also suc. x,
 (iii) if it contains suc. x it contains also x,
then every integer belongs to the class u.

These postulates define an algebraic field, whose nature is at once seen to agree with that of the natural field, suc. x being interpreted as $1 + x$, and sym. x as $-x$. M. Padoa did not get beyond this definition, possibly because he had entered so minutely into the details of the proof of the independence of the seven postulates that he had exhausted his allowance of time.

A great part of the Friday morning sitting of these two sections was devoted to the discussion of a resolution, proposed by M. Leau, in favor of the adoption of some special artificial language as the vehicle for all scientific communications. Though no particular language was named in the resolution, it was made clear that "Esperanto" was the language intended. Its advocates, MM. Leau, Padoa, Boccardi, Laisant, and others, disclaimed any wish to substitute it for natural languages, but urged its adoption as the vehicle for international intercourse; this view they upheld with great earnestness "on behalf of humanity," as M. Laisant put it. The opposite view was upheld with equal earnestness, if less vehemence, by MM. Schroeder, Vassilief, Maggi, and others, chiefly on the ground that any such language is entirely unnecessary; as M. Maggi remarked, mathematics already has a universal language, the language of formulæ. In the end the suggestion of M. Vassilief was adopted, that the Congress should place itself on record as opposed to any unnecessary diversity in the languages employed, that is, practically, to the use of any language for scientific purposes other than English, French, German, and Italian, though these languages were not specified in the resolution adopted.

On Saturday, August 11th, the concluding general meeting was held at 9 a. m. The first business was to determine the time and place for the next meeting. At Zurich, Professor Klein, on behalf of the German Mathematical Society, had expressed their great desire that the third congress should be held in Germany; and a definite invitation to this effect was now laid before the Congress, and unanimously accepted. The place of meeting will probably be Baden-Baden; the date decided upon is 1904, and the time is to be either at the beginning or the end of the summer vacation. No other business was transacted, and the two general addresses appointed for the day were then delivered by MM. Mittag-Leffler and Poincaré. Immediately after the conclusion of the President's address, he dismissed the Congress with the words "La séance est levée, le congrès est clos."

M. Mittag-Leffler's address was entitled "Une page de la vie de Weierstrass"; in this he considered in some detail Weierstrass's attitude towards some of the mathematical ideas of his time, illustrating by copious extracts from his correspondence; unfortunately it is not possible to give any adequate account of it. M. Poincaré's can be given more fully.

H. POINCARÉ: *Du rôle de l'intuition et de la logique en mathématiques.*

It is obvious that there are two entirely different types of mind among mathematicians, manifesting themselves in two distinct methods of treating mathematical questions. Those of the first type are dominated by logic; those of the second are guided by intuition. They may be called analysts and geometers, though it is not really a question of the subject with which they deal; the analyst remains an analyst even when working at geometry, and the geometer employing himself on pure analysis is still a geometer. Nor is the distinction a mere matter of education ; a man is born a mathematician, he does not become one; and either he is born an analyst or he is born a geometer. The two types of mind are equally necessary for the progress of the science ; each has accomplished great things that would have been impossible to the other.

At first sight the ancients seem to have all been intuitionalists, but this impression disappears on closer study. Euclid, for instance, was a logician, even though every stone of his edifice is due to intuition. The natural tendencies have not changed, only their manifestation. There has been an evolution, due to the increasing recognition of the fact that intuition cannot give rigor, nor even certainty ; a proof that relies on concrete images may be very deceptive. It was soon realized that rigor cannot be expected in the demonstrations unless it is to be found in the definitions ; so long as the objects of reasoning were given simply by the bodily senses or the imagination, there was no precise idea on which reasoning could be based. Thus the efforts of the logicians were concentrated on the definitions, one result of which is that mathematics has become arithmetized.

The question arises, is this evolution ended—have we at last attained to absolute rigor, or do we deceive ourselves as our fathers did? Philosophers tell us that it is impossible to eliminate intuition altogether from our reasonings, for no science can spring from pure logic alone. To designate this other essential, we have no name but intuition ; but this covers many different ideas. There is (1) the appeal to the bodily senses and to imagination ; (2) generalization by induction ; (3) the intuition of pure number ; on this last a veritable mathematical method is based, while from the first two no certainty can be derived. The analysis of the present day constructs its demonstrations solely from syllogisms and this intuition of pure number ; we may say that at last absolute rigor is attained.

The philosophers now object that what has been gained in rigor has been lost in actuality ; the approach toward the logical ideal has been secured by cutting the ties with reality. For the sake of the demonstration a mathematical definition is substituted for the object, and it still remains to prove that the concrete reality answers to the definition. But as this is an experimental truth, it is not the business of mathematics to establish it. It is a great step forward to have separated these two things; nevertheless there is something in the philosophic objection. In becoming rigorous, mathematics has assumed a certain character of artificiality; if it is clear how questions can be resolved, it is no longer clear how and why they arise. We seek for reality; but this does not reside in the separate steps of the demonstration; it must be sought rather in the something that makes for unity. The microscopic examination of an elephant gives no idea of the animal itself; the fairy-like structure of silicious needles which is all that is left of certain sponges cannot be understood without reference to the living sponge by which this form was imposed on the silicious particles. Logic by itself cannot give the view of the whole which is indispensable alike to the inventor and to him who desires really to understand the inventor. Logic, which alone gives certainty, is simply the instrument of demonstration; the instrument of discovery is intuition.

But analysts also are inventors; hence they cannot always be proceeding from the general to the particular, as the rules of formal logic demand, for scientific conquests are made only by generalization. There is however a perfectly rigorous process, that of mathematical induction, by which it is possible to pass from the particular to the general.* For the profitable use of this, to recognize the analogies whose presence makes it applicable, the analyst must have the direct feeling for the unity of an argument, for its soul and spirit; for him the most abstract entities must be living beings. What is this but intuition? This however does not invalidate the distinction already drawn, for it is an intuition entirely different in nature from the sensible intuition founded in imagination alone, even though psychologists may finally pronounce it also to have a sensual foundation. It is the intuition of pure logical form, which together with the intuition of pure number makes not only demonstration, but also discovery, possible to the analyst. Thus among the analysts inventors do exist, but not many; it remains true that the most usual instrument of invention in mathematics is sensible intuition.

On the evening preceding the formal opening of the Congress an informal reunion of the members, about half of whom were present, was held at the Café Voltaire. On Tuesday afternoon, after the rising of the sections, the members were entertained at the École normale supérieure. At noon on Sunday, August 12th, a very successful banquet was held at the Salle de l'Athénée-Saint-Germain, when, in the absence of M. Poincaré, M. Darboux presided very pleasantly. Toasts to those present, to the hosts, to the absent, to M. Darboux, and to the next Congress, were proposed by MM. Darboux, Geiser, J. Tannery, Stephanos, and Vassilief. Invitations to receptions held by the President of the Republic and by Prince Roland Bonaparte were accepted by a number of the members.

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* Poincaré, "Sur la nature du raisonnement mathématique," *Revue de métaphysique et de morale*, vol. 2 (1894), pp. 371–384.

A very large attendance had been expected, on account of the additional attractions offered by the Exhibition; and the answers to the circulars first sent out went far to justify this anticipation, for up to last December about 1,000 mathematicians had signified their intention of being present, with 680 members of their families. The membership fee was fixed at 30 francs, with an additional 5 francs for every member of the family. As a matter of fact, the total attendance can hardly have exceeded 250 in all. There seems very little doubt that a large proportion were kept away by distaste of the crowds that were supposed to be visiting the Exhibition, and by the rumored difficulty in obtaining accommodation, a difficulty that seems to have existed mainly in the circulars of the various agencies; but the great heat of July certainly decided many to stay away who would otherwise have been present.

The countries represented were as follows: France, 90; Germany, 25; United States, 17; Italy, 15; Belgium, 13; Russia, 9; Austria, 8; Switzerland, 8; England, 7; Sweden, 7; Denmark, 4; Holland, 3; Spain, 3; Roumania, 3; Servia, 2; Portugal, 2; South America, 4; with single representatives from Turkey, Greece, Norway, Canada, Japan, Mexico. This list is only approximate, as no revised list of members was issued. Among the members from the United States were Professors Allardice, E. W. Brown, Dickson, Ely, Hagen, Halsted, Hancock, Harkness, Keppel, Lovett, Macfarlane, Pell, Scott, Stringham, Webster.

While the four languages, English, French, German, and Italian, were admitted on equal terms, by the constitution of the congresses, the great preponderance of French was noticeable. At Zürich, this preponderance existed in friendly intercourse, but French and German were about equally used in the communications, whereas in Paris all the general addresses, and most of the sectional papers, were in French, possibly out of compliment to our hosts. Probably at Baden-Baden French and German will be used in about equal proportions as at Zürich.

The distribution of the authors of communications among the different countries may be of interest. The four general addresses were delivered by representatives from France, Italy, Germany, Sweden; including these, papers were read by members from France, 13; Italy, 9; United States, 6; Germany, 4; Sweden, 4; Austria, 2; England, Greece, Holland, Japan, Portugal, Russia, Servia, and Spain, one each. This list may require some slight modification, as the only programme issued needed some corrections, but it is substantially correct. In some cases two or three communications were made by one member, thus bringing up the number of sectional papers to about 50; some of these were presented by title only in the absence of their authors. The general meetings occupied four hours, the sectional meetings 26. About 200 members were present at each of the general meetings; at the sectional meetings the average attendance was about 90, and as two sections were usually sitting at the same time, this accounts very fairly for the members. About 160 were present at the banquet.

The arrangements excited a good deal of criticism. The committee of organization had doubtless special difficulties to contend with, as M. Laisant, to whom the secretarial part had been assigned, was unable to undertake it owing to the pressure of other duties. The mistake was then made of entrusting a part of this responsibility to the firm of Carré and Naud, whereas in such a case personal interest and individual responsibility are indispensable for ensuring proper attention to the various details of organization. Owing to this, members arriving in Paris had very great difficulty, during the first two days, in obtaining the necessary information in time for it to be of any service. The want of a common assembly room, where members might conveniently meet one another with or without concerted arrangement, was seriously felt. The arrangements in Zürich were so admirably complete in every point, that these defects were even more conspicuous by comparison. There is no doubt that a smaller town lends itself best to such a gathering; it is not so much that there is less division of interests, as that the members are more in evidence, and so have a better chance of *realizing* one another. The first object of an international congress of mathematicians is to enable its members to meet one another in circumstances that shall excite the mathematical faculty, and make manifestations of mathematical interest most natural, thus encouraging the exchange of ideas both between individuals and larger groups. The mathematician who is in any degree a specialist is in general rather solitary in the average college—he would have been better off in Noah's Ark, for at the worst there would have been two of a kind. For his mind's health it is well that he should occasionally be thrown with those of kindred interests; it is well too that he should be made to feel the unity of mathematics.

The general addresses are of course under one aspect the most important part of the formal proceedings, giving broad views of the whole subject, and helping the pronounced specialist to realize his affiliations with other regions. Did a congress achieve no other end than that of evoking such addresses as those of Klein, Hurwitz, Cantor, Poincaré, Volterra, at Zürich and Paris, and presenting them to mathematicians of such widely differing interests, its existence would be amply justified. Something more might be done in a similar direction as regards the sectional meetings; an address, not necessarily by the president of the section, dealing in a broad manner with the region, or some part of the region, assigned to the section, might very well be arranged for. While the general addresses enable the individual to appreciate the relation of his special subject to mathematics as a whole, these sectional addresses would assist him in the equally important and even more difficult task of gauging his own relation to his special subject. One or two of the communications offered, both at Paris and at Zürich, were of this nature; but the matter should not be left to chance, it is well worth systematically arranging. Such an address would deal sometimes with the general ideas of the subject, sometimes with what remained to be done, sometimes with what had been accomplished during the last few years.

As to the nature of the more special papers, it hardly seems advisable to restrict the present liberty, not even to the extent that is found salutary in the regular meetings of a mathematical society; encouragement is perhaps more needed. But some control over the time consumed should be exercised; not only a theoretical control, confined to a printed statement that it must not exceed twenty minutes, but a real control, exerted as a regular thing. Ten or fifteen minutes, well employed, is quite long enough for the ordinary type of special theorem, for an outline of the method of proof is sufficient in an oral communication, and another five or ten minutes ought to enable the speaker to indicate its connections within the subject. The strict limitation to a time previously determined would in most cases be beneficial to the author, obliging him to select, subordinate, and group his details. It is tolerably certain that if the author regards all details as equally important, his auditors will regard all as equally unimportant.

One thing very forcibly impressed on the listener is that the presentation of papers is usually shockingly bad. Presumably the reader desires to be heard and understood; to compass these ends, instead of speaking to the audience, he reads his paper to himself in a monotone that is sometimes hurried, sometimes hesitating, and frequently bored. He does not even take pains to pronounce his own language clearly, but slurs or exaggerates its characteristics, so that he is often both tedious and incomprehensible. These failings are not confined to any one nationality; on the whole the Italians, with their clear and spirited enunciation, come nearest to being free from them. It would be invidious and impertinent to mention names; the special sinners sit in both high and low places. But it is perhaps pardonable to refer to M. Mittag-Leffler's presentation of his paper to Section II as showing in how admirable and engaging a style the thing can be done. It is not given to everyone to do it with this charm; but there is no excuse for any normally constituted human being, sufficiently versed in mathematics, failing to interest a suitable audience for a reasonable time in that which interests himself, always provided that it be of sufficient novelty either in matter or in mode of treatment to justify him in presenting it at all.

At the Zürich Congress certain matters were energetically discussed in Section V; extensive support was then given to resolutions in favor of constituting permanent commissions charged to consider 1° general reports on the progress of mathematics, 2° matters of bibliography and terminology, 3° the possibility of giving some permanent character to the Congress, by means of a central bureau or otherwise. Though these resolutions were not voted upon directly, it being felt that they required more deliberate discussion, yet at the concluding general meeting the members of the Zürich bureau were appointed a commission to consider the questions that seemed of most importance, and to furnish the Mathematical Society of France with such information on these points as might be useful in preparation for the Congress of 1900. At the joint sitting of Sections V and VI in Paris M. Dickstein asked a question on behalf of the members, namely, was not the Congress to hear anything of the deliberations of this commission? No satisfactory answer was forthcoming; M. Laisant replied simply that the Mathematical Society of France had been so taken up with material preparations for the Congress that it had not been able to enter upon any of these matters. He took the opportunity, however, of directing attention to the *Annuaire des Mathématiciens*, projected by Carré and Naud, as carrying into effect one suggestion made at Zürich. The question then dropped, but it was felt that this left matters in a very unsatisfactory state. It is to be hoped that a different report will be given in 1904, that the members of the Baden congress will not simply hear papers and meet friends, but have a chance to consider these matters of international concern. Some questions of business arise in every science; they tend to settle themselves by a kind of tentative process, a survival of the fittest, or rather by general tacit consent. This is often the best process; any attempt at forcing an expression of general agreement may result in checking development by encouraging too early a crystallization. But there are some matters, depending on concerted action, that are ripe for decision, and that cannot profitably be settled by any one nation for itself; matters in which for want of a general agreement labor may be wasted. Such matters may naturally be decided by an international congress, whose decisions will simply have the force of general consent. One such question is that of a classification of mathematical sciences. At least two well-known systems are in use, and there may be others. Multiplication of systems of classification, like the multiplication of universal languages, practically destroys the good of any and all; the congress ought to pronounce in favor of some one. As to the preparation of special reports, it seems doubtful whether this will be done best by the congress at present. In course of time it may assume academic functions and responsibilities, but it will be necessary for it to prove its continuity before it can with any propriety expect to control mathematical efforts. Encouragement and recognition would seem to be its appropriate province at present in these respects. The question as to how this continuity is to be obtained is a rather serious one, and deserving of careful discussion. A central bureau with various functions was suggested when the matter was under discussion at Zürich; but there are objections to this. If the organizers of each congress will make it a point of honor to act on the recommendations of the preceding congress, taking into consideration the resolutions passed and doing what can be done towards carrying them into effect, possibly sufficient continuity may be attained without the red tape that would coil itself about any permanent bureau. Some questions are better left undecided. International agreement is not wanted on all points; international rivalry and emulation still have their part to play, helped by the international friendships that are promoted by such gatherings as these international congresses.

In conclusion, I must express my thanks to many of those whose names appear in this report for the assistance I have received from them in its preparation.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
October, 1900.

THE FORTY-NINTH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE forty-ninth annual meeting of the American Association for the Advancement of Science, which was held at Columbia University June 23–30, was from the point of view of scientific work one of the most successful in the history of the Association. Sixteen affiliated societies met with the Association and contributed greatly to the importance and interest of the meeting. Two of these, the American Mathematical Society and the Astronomical and Astrophysical Society, held joint sessions with Section A.

The next meeting of the Association will be held in Denver during the last week of August, 1901, under the presidency of Professor Minot of the Harvard Medical School. Professor James McMahon, Cornell University, will be vice-president of Section A, and Professor H. C. Lord, Ohio State University, will be secretary. Forty-one out of the total number of forty-nine annual meetings of the Association have been held during the month of August, while the recent New York meeting was the first that was held in June. The next meeting will be farther west than hitherto, but it seemed to be the general opinion that this was desirable in order to extend the influence of the Association. Pittsburg was recommended as the place of meeting in 1902.

The meetings of the section of mathematics and astronomy were well attended. The officers of this section were: vice-president, Asaph Hall, Jr.; secretary, W. M. Strong; councilor, Ormond Stone; sectional committee, Alexander MacFarlane, J. F. Hayford, C. L. Doolittle, W. M. Strong, Harold Jacoby, Edgar Frisby, W. S. Eichelberger. The Council elected the following mathematicians and astronomers to fellowship in the Association: E. S. Crawley, G. A. Hill, W. J. Humphreys, H. C. Lord, L. H. Orleman, Mary W. Whitney, P. S. Yendell.

The opening address of the Vice-President of Section A, "On the teaching of astronomy in the United States," has been published in *Science*, July 6, 1900, pp. 15–20. In the absence of the Vice-President, Simon Newcomb, C. L. Doolittle and Ormond Stone presided at the meetings of this Section. The following list of papers read before the Section does not include those of the American Mathematical Society and of the Astronomical Society which were read at the joint sessions of these societies and Section A.

(1) President H. S. PRITCHETT: "The functions, organization, and future work of the U. S. Coast and geodetic survey."

(2) Mr. J. F. HAYFORD: "The precise-level net of the U. S. and a new levelling instrument."

(3) Professor T. J. J. SEE: "The propagation of the tide wave."

(4) Professor T. J. J. SEE: "The dimensions and density of Neptune."

(5) Professor E. FRISBY: "Some remarkable properties of recurring decimals."

(6) Professor G. T. SELLEW: "History of the complex number."

(7) Professor JAMES MCMAHON: "The expression of a rational polynomial in a series of Bessel Functions of the nth order."

(8) Dr. G. A. MILLER: "Report on the groups of an infinite order."

(9) Professor L. E. DICKSON: "Definition and examples of Galois fields."

(10) Mr. W. B. FITE: "On the metabelian groups whose invariant operators form a cyclical subgroup."

(11) Mr. H. E. HAWKES: "Note on Benjamin Pierce's linear associative algebra."

(12) Mr. F. H. LOUD: "Sundry metrical theorems connected with a special curve of the fourth class."

(13) Mr. G. A. HILL: "Variations of latitude."

(14) Mr. G A. HILL: "The comparative accuracy of the transit circle and the vertical circle."

(15) Professor J. N. STOCKWELL: "New light on ancient eclipses."

(16) Professor J. N. STOCKWELL: "Secular variations of the motions of the planets."

(17) Professor J. N. STOCKWELL: "A new method of computing the Laplace coefficients of planetary pertubation."

(18) Miss MARY PROCTOR: "Miss Catharine Wolf Bruce."

(19) Miss M. E. TRUEBLOOD: "The directive force of philosophy upon mathematics."

(20) Mr. W. G. LEVISON: "On a method of photographing the entire corona, employed at Newberry, S. C., for the total solar eclipse, May 28, 1900."

Two of the titles that appeared on the programme are not reproduced here since these papers did not reach the secretary in time to be presented at the meeting. In the absence of their authors the papers by Professor McMahon and Mr. Fite were presented by Professor Crawley and Dr. Miller respectively. Mr. Hawkes' paper was read by title. The two papers by Mr. Hill will be published in the *American Journal of Science* and in *Science* respectively. Dr. Miller's report will appear in the BULLETIN. Abstracts of the other papers, with the exception of the last three, are given below.

President H. S. Pritchett, lately Superintendent of the U. S. Coast and geodetic survey, considered the three questions: "What is the purpose of this service? Is it properly organized to carry out this purpose? What lines of work should be followed to accomplish the purpose in view?" The first and principal work of the Coast and geodetic survey is the hydrographic survey of the coasts of the United States and the islands under her jurisdiction. In hydrography and topography a part of the energy of the service will go to the resurvey of parts of the mainland of the United States; but by far the greater effort will go to the hydrographic surveys of Porto Rico, Hawaii, the Philippine Islands, and Alaska. In geodesy, the most interesting work of the next eight years will be the completion of an arc extending along the ninety-eighth meridian from the Rio Grande to the Canadian border. By the coöperation of Mexico and of Canada it is expected that this arc will ultimately have for its southern limit a latitude of about 19 degrees and for its northern terminus the latitude of nearly 70 degrees. The completion of the precise level net for the United States, and the differential gravity determinations now being carried out in connection with similar work by other nations, are the most important other objects of this service. The work of the Division of terrestrial magnetism contemplates a general magnetic survey of the country and of the waters adjacent within the next ten years.

Mr. Hayford's paper is in abstract as follows: The net of precise level lines recently adjusted in the Computing Division of the Coast and geodetic survey comprises over 12,000 miles of spirit leveling, and 1,400 miles of water leveling, run by the Coast and Geodetic Survey and other organizations. The manner of making several preliminary "studies" before making the final least square adjustment was explained. The points requiring especial attention were, the assignment of relative weights to different classes of leveling, the attempt to detect systematic errors, and the assignment of the relation of weight to length of line. The least square method of computation was used throughout and the assumptions upon which it is based were checked, as far as possible, by direct appeals to the facts. The only well-marked form of a systematic error found, is one which is due to changes of temperature in the instrument during the period of observation at a station. The instrument shown was one which is about to be put into use in the Coast and geodetic survey; it is designed with special reference to eliminating the systematic error just referred too, and to furnish rapid observations of the highest degree of accuracy.

The object of the first paper by Professor See, was to direct attention to certain points in the theory of the propagation of the tide wave. After defining the tide wave by the general relations for horizontal and vertical motion,

$$x' = x + \varphi(x_1 t), \quad y' = y + \psi(\ _1 t),$$

which are transformed into

$$\varXi = A\left(e^{\frac{2\pi\gamma}{\lambda}} + e^{-\frac{2\pi\gamma}{\lambda}}\right)\cos(nt - \nu x),$$

$$H = -A\left(e^{\frac{2\pi\gamma}{\lambda}} - e^{-\frac{2\pi\gamma}{\lambda}}\right)\sin(nt - \nu x),$$

γ being the depth of the sea, λ the wave length, and ν any integral number, the author shows that in case of the tides, where λ is large compared to γ, the path of any particle is an elongated ellipse, giving a horizontal motion about 1,000 times greater than the vertical motion. He then gives the tide-generating potential in the usual form

$$V = \frac{3}{2}\frac{m\nu^2}{\rho^3}\left(\cos^2\omega - \frac{1}{3}\right),$$

and shows by derivation the lengths of the four arcs into which the circumference is divided, two being each 109°.5 in length, the other two 70°.5. A figure showing the motion of the water in every part of an equatorial canal at any instant, indicates clearly why the wave advances when once started, as the flow is forward in the two long arcs, and backward in the short arcs. The author states that the tide wave is at first a forced wave, afterwards acting as a free wave ; but this view of the propagation holds for both classes of oscillations, when treated separately, and is approximately true when the two classes of waves are compounded.

In his second paper Professor See treats historically the question of the diameter of Neptune, showing that the earliest measures, made in 1846, place it at about 3″, and that subsequent measures range from 4″.3 to 2″.00. With every improvement in the definition of telescopes, the diameter has shrunk. He then showed, from observations made in 1899–1900 with the great equatorial of the Naval Observatory under better conditions than have been enjoyed by previous observers, that the true diameter of the planet is about 2″.00, or 27,190 miles. The diameter hitherto accepted in standard works on astronomy is 34,000 miles. The former density was 1.11 that of water, but this is now doubled and put at 2.26.

Professor Frisby gave some brief special methods of combining numbers, and especially of finding the recurring decimals which represent rational fractions.

Professor Sellew traced the history of the complex number from the first appearance of the imaginary in the Stereometrica of Hero of Alexandria in the second century B. C. The idea that the imaginary must be in the plane without the line of real numbers was first given by Wallis (1685). Kühn's construction (1753) introduced the idea of direction. Wessel (1797) was the real inventor of the modern representation of the complex number. His analytic representation of direction was the best before the time of Hamilton.

It is known that in certain cylindrical boundary problems it is necessary to expand $f(r, \theta)$ in a Fourier series of the form

$$\sum_{n=1}^{\infty} \varphi_n(r) \cos n\theta,$$

and then to expand $\varphi_n(r)$ in an nth order Bessel series of the form

$$A_1 J_n(\lambda_1 r) + A_2 J_n(\lambda_2 r) + \cdots,$$

in which $\lambda_1, \lambda_2, \cdots$ are determined from the roots of a certain equation to be satisfied on the boundary $r = a$, and the coefficients are given by

$$A_s = \frac{\int_0^a \varphi_n(r) . r J_n(\lambda_s r) dr}{\int_0^a r [J_n(\lambda_s r)]^2 dr}.$$

The value of the definite integral in the denominator is given in the text-books, but no method of evaluating the integral in the numerator has been given. The object of the paper by Professor McMahon is to reduce this integral in the commonest case, viz., that in which $\varphi_n(r)$ is expressible in a series of powers of r with integer exponents (positive or negative). For this purpose formulæ are given by means of which $\int x^m J_n(x) dx$ is made to depend on $\int x J_n(x) dx$ and $\int J_n(x) dx$, and ultimately on $\int J_0(x) dx$ alone.

Thus the complex numerical treatment of a large class of cylindrical harmonic problems requires the introduction and tabulation of a new fundamental transcendent, which may be defined by the equation

$$H_0(x) = \int_0^x J_0(x) dx.$$

This function may be tabulated either from the power-series, or from the relation

$$H_0(x) = 2[J_1(x) + J_3(x) + J_5(x) + \cdots].$$

Professor Dickson developed the definition of Galois field as follows: Let p be a prime number and $P(x)$ a polynomial of degree n in x having integral coefficients, and suppose $P(x)$ to be *irreducible* modulo p, *i. e.*, suppose it to be impossible to give $P(x)$ the form

$$P(x) = P_1(x) \cdot P_2(x) + p \cdot P_3(x),$$

where the $P_i(x)$ are polynomials having integral coefficients, the degrees of $P_1(x)$ and $P_2(x)$ being $<n$. If we divide any polynomial $F(x)$ having integral coefficients, by the function $P(x)$, we obtain a quotient $Q(x)$, and a remainder which can be written

$$f(x) + p\ \ q(x), \quad f(x) \equiv a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-2},$$

where every a_j is a positive integer $<p$. Hence

$$F(x) \equiv f(x) + P(x)\,Q(x) + p \cdot q(x).$$

We say $f(x)$ is the residue of $F(x)$ modulo p and $P(x)$ and write

$$F(x) \equiv f(x) \quad [\text{mod}\ p, P(x)].$$

The totality of functions $F(x)$ obtained by giving to the polynomials $Q(x)$ and $q(x)$ all possible forms is said to constitute a class of residues. Two polynomials

$$F_i(x) = f_i(x) + P(x)\,Q_i(x) + p.q_i(x) \qquad (i = 1, 2)$$

belong to the same or different classes according as f_1 and f_2 are identical or not. Each a_j in $f(x)$ having p values, there are evidently p^n distinct classes of residues. The class to which $F_1 \pm F_2$ or $F_1\ F_2$ belongs depend merely upon the functions $f_1 \pm f_2$ or $f_1\ f_2$, being independent of the Q_i and q_i. Hence classes of residues combine unambiguously under addition, substraction and multiplication. Furthermore, an arbitrary class C_{F_2} may be divided by any class C_{F_1}, not the class zero, yielding as quotient a unique third class. In fact, p being prime and $P(x)$ irreducible modulo p, and $F_1(x)$ being not divisible by $P(x)$ modulo p, we can determine, by a process analogous to that for finding the greatest common divisor, integral functions $F_1'(x)$ and $P'(x)$ such that

$$F_1'(x) \cdot F_1(x) - P(x) \cdot P'(x) \equiv 1 \quad (\text{mod}\ p).$$

Hence the function $F_1'F_1$ belongs to the class unity. It follows that

$$\frac{C_{F_2}}{C} = \frac{C_{F_2}C_{F_1'}}{C_{F_1}C_{F_1'}} = C_{F_2F_1'}.$$

The p^n classes of residues modulis p and $P(x)$ therefore form a field, called a Galois field of order p^n.

The main object of Mr. Fite's paper was to demonstrate the following theorems :

THEOREM I. If the commutators of a metabelian* group G of order p^m, p being a prime, form a cylical group of order p^α, then the number of independent generators of any given order of the group of cogredient isomorphisms G' must be even.

THEOREM II. There are exactly $\beta + 1$ groups of order $p^{\alpha+2\beta s}$, p being an odd prime, s being any positive integer, and $\beta \leqq \alpha$, such that their invariant operators form a cylical group of order p^α and that their groups of cogredient isomorphisms are abelian of the type (β, β, ⋯ to $2s$ terms).

THEOREM III. The number of metabelian groups whose invariant operators form a cyclical group of order p^α and whose groups of cogredient isomorphisms are of the type (α_1, α_1, ⋯, to $2s_1$ terms; α_2, α_2 ⋯ to $2s_2$ terms ; ⋯ ; α_n, α_n, ⋯ to $2s_n$ terms), where s_1, s_2, ⋯, s_n are positive integers and $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_{n-1} < \alpha_n$, is $(\alpha_1+1)(\alpha_2 - \alpha_1 + 1)(\alpha_3 - \alpha_2 + 1)$

$$\cdots(\alpha_n - \sigma_{n-1} + 1).$$

The curve studied by Mr. Loud is the parallel to the hypocycloid of four cusps, and is briefly noticed by Salmon in his Higher plane curves, §§ 118 and 119. The present paper gives a simple method of construction by points and establishes several theorems connecting the curve with the metric geometry of n lines in a plane. The treatment adopted employs the circular coördinates x and y, equal respectively to $x + iy$ and $x - iy$ in the Cartesian system. Some of the more obvious properties of the curve, as well as its varieties of form, are deduced. The methods and results given by Professor Morley in the second number of the *Transactions* have been extensively employed.

In his first paper Professor Stockwell argues that ancient eclipses throw much more light on ancient chronology than might be inferred from the paper by Professor McFarland which was read before this Section at the Columbus meeting. The second paper is intended to show that there are secular equations of the mean distances of the planets from the sun. The third paper is in abstract as follows :

The Laplace coefficients of planetary perturbation arise from the development of the function

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*If the commutators of a group are self-conjugate the group is called *metabelian*. The group of cogredient isomorphisms of such a group is abelian, and conversely.

$$\{1 + a^2 - 2a \cos \beta\}^{-s}$$

in an infinite series depending on the cosines of β and its multiples. If we assume that

$$\begin{aligned}\{1 + a^2 - 2a \cos \beta\}^{-s} = \tfrac{1}{2}b_s^{(0)} + b_s^{(1)} \cos \beta + b_s^{(2)} \cos 2\,\beta \\ + b_s^{(3)} \cos 3\beta + \cdots + b_s^{(i)} \cos i\beta,\end{aligned} \tag{1}$$

the coefficients $b_s^{(i)}$ are the Laplace coefficients required.

If we put
$$\frac{2a}{1 + a^2} = a', \tag{2}$$
we shall have

$$\{1 + a^2 - 2a \cos \beta\}^{-s} = (1 + a^2)^{-s} \{1 - a' \cos \beta\}^{-s}; \tag{3}$$

and if we now assume that

$$\begin{aligned}\{1 - a' \cos \beta\}^{-s} = \tfrac{1}{2} \dot{b}_s^{(0)} + \dot{b}_s^{(1)} \cos \beta + b_s^{(2)} \cos 2\beta \\ + b_s^{(3)} \cos 3\beta + \cdots + b_s^{(i)} \cos i\beta\,,\end{aligned} \tag{4}$$
we shall have
$$b_s^{(i)} = b_s^{(i)} \cdot \{1 + a^2\}^{-s}. \tag{5}$$

If we now develop the first member of equation (4) by the binomial theorem, we shall have an infinite series of terms which may readily be computed by means of a table of logarithms of the coefficients of the different powers of cos β, together with a similar table of logarithms of the coefficients $\cos^n\beta$ when it is developed in terms of the cosines of β and its multiples.

Tables containing the logarithms of the coefficients of these functions have been computed, so that we may easily compute all values of $\dot{b}_s^{(i)}$ for all values of $i = 0$, to $i = 35$. And when $\dot{b}_s^{(i)}$ has been found $b_s^{(i)}$ may be found by equation (5).

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NOTE ON GEOMETRY OF FOUR DIMENSIONS.

BY PROFESSOR E. O. LOVETT.

(Read before the American Mathematical Society, April 28, 1900.)

1. Speculations relative to the geometry of n dimensional space havé followed several fairly well-defined trends which not infrequently cross one another: 1° A direct extension of the Cartesian geometry, which extension is to be regarded as nothing more than a convenient form of phraseology; in this form did n dimensional space spring forth from the minds of Grassmann, Cayley, Gauss, and Cauchy, and the idea was likely familiar to Euler and Lagrange. 2° The transformation of the ordinary visualizable spaces of two and three dimensions into manifoldnesses of higher or lower dimensions by introducing space elements other than the point or its dual element; for example, the line geometry of Plücker, the sphere geometry of Lie, the five dimensional manifoldness of all conics in the plane as an auxiliary to Ball's theory of screws; this category becomes more concrete perhaps than any other. 3° The absolute geometry of space; here would appear the celebrated dissertation of Riemann, the well-known memoirs of Helmholtz and Lie and the elaborate treatise of Veronese. 4° The extension of the methods of ordinary differential geometry to spaces of many dimensions; to this class belong the works of Christoffel, Beltrami, Bianchi, Cesàro, and Ricci, and the quite recent contributions of Darboux and his pupils. 5° The direct extension of the concepts and problems of metrical and projective geometry of ordinary space, as exemplified in the memoirs of Jordan, d'Ovidio, and Veronese. 6° The theory of birational correspondences between n dimensional aggregates as studied by Noether, Kantor, and Brill. 7° The descriptive geometry of space of n dimensions as begun in the papers of Veronese, Stringham, Schlegel, and Segre. 8° The kinematics of higher spaces as developed by Jordan, Clifford, and Beltrami. 9° The interpretation of n dimensional geometry in the light of the theory of groups as exhibited by Lie, Klein, and Poincaré.

2. It is proposed here to make an expository contribution to the ninth and fourth of the above categories, constructing ordinary four dimensional space by the method of Lie's theory of continuous groups, and studying curves of triple

curvature by the intrinsic analysis developed by Cesàro in his Lezioni di geometria intrinseca.

3. With regard to the space it is assumed:

1° That it is a four dimensional manifoldness, *i. e.*, that four independent data are necessary and sufficient to determine the position of an element of the manifoldness; these four independent things are called the coördinates of the element.

2° That a figure of the manifoldness possesses ten degrees of freedom within the manifoldness; *i. e.*, that ten independent data are necessary and sufficient to render a rigid body fixed in position; the latter ten independent things are called the parameters of the figure.

For convenience let the element be called a point, and its coördinates be designated by x, y, z, t. Consider any figure containing this point and let its parameters be $a_1, a_2, \cdots, a_{10}$. Let the figure assume a new position and call x', y', z', t' the coördinates of the new position of (x, y, z, t). Then

$$x' = \xi(x, y, z, t, a_1, a_2, \cdots, a_{10}), \quad y' = \eta(x, y, z, t, a_1, a_2, \cdots, a_{10}),$$
$$z' = \zeta(x, y, z, t, a_1, a_2, \ldots, a_{10}), \quad t' = \tau(x, y, z, t, a_1, a_2, \cdots, a_{10}).$$

The operation changing x, y, z, t into ξ, η, ζ, τ represents one of the motions of a four dimensional figure, and the ensemble of all these operations constitutes a continuous group with ten parameters. The identical transformation ought to appear among these operations. There must then be a system of the parameters $a_1, a_2, \cdots, a_{10}$ such that

$$\xi = x, \quad \eta = y, \quad \zeta = z, \quad \tau = t.$$

It is legitimate to assume the preceding system of values to be

$$a_1 = a_2 = \cdots = a_{10} = 0.$$

An infinitesimal transformation of the group is one whose parameters differ by infinitesimals from those giving the identical transformation; in this case the infinitesimal transformation is obtained by assigning infinitesimal values to the parameters themselves; *i. e.*, by such a transformation x, y, z, t are changed to

$$x + \sum_1^{10} a_i \frac{\partial \xi}{\partial a_i}, \quad y + \sum_1^{10} a_i \frac{\partial \eta}{\partial a_i}, \quad z + \sum_1^{10} a_i \frac{\partial \zeta}{\partial a_i}, \quad t + \sum_1^{10} a_i \frac{\partial \tau}{\partial a_i},$$

in the partial derivatives of which the a's should be put equal to zero.

Writing for short

$$p \equiv \frac{\partial f}{\partial x}, \quad q \equiv \frac{\partial f}{\partial y}, \quad r \equiv \frac{\partial f}{\partial z}, \quad s \equiv \frac{\partial f}{\partial t},$$

Lie's symbol for the above infinitesimal transformation is

$$I \equiv p \sum_1^{10} a_i \frac{\partial \xi}{\partial a_i} + q \sum_1^{10} a_i \frac{\partial \eta}{\partial a_i} + r \sum_1^{10} a_i \frac{\partial \zeta}{\partial a_i} + s \sum_1^{10} a_i \frac{\partial \tau}{\partial a_i}.$$

Putting

$$J_i \equiv p \frac{\partial \xi}{\partial a_i} + q \frac{\partial \eta}{\partial a_i} + r \frac{\partial \zeta}{\partial a_i} + s \frac{\partial \tau}{\partial a_i}, \qquad (i = 1, 2, \cdots, 10)$$

any infinitesimal transformation is written

$$I \equiv \sum_1^{10} a_i J_i.$$

By a fundamental theorem of Lie, if we put

$$(J_i, J_j) \equiv \frac{\partial J_i}{\partial p}\frac{\partial J_j}{\partial x} - \frac{\partial J_i}{\partial x}\frac{\partial J_j}{\partial p} + \cdots + \frac{\partial J_i}{\partial s}\frac{\partial J_j}{\partial t} - \frac{\partial J_i}{\partial t}\frac{\partial J_j}{\partial s},$$

then

$$(J_i, J_j) = \sum_{k=1}^{k=10} \overset{(i,j)}{\lambda_k} J_k, \quad (i = 1, \cdots, 10\,;\ j = 1, \cdots, 10) \qquad (1)$$

where the λ's are constants. There are forty-five of these equations, but the four hundred and fifty λ's are not wholly arbitrary, since the following one hundred and twenty identities of Jacobi must hold

$$\begin{gathered}(J_i, (J_j, J_k)) + (J_j, (J_k, J_i)) + (J_k, (J_i, J_j)) = 0, \\ (i, j, k = 1, 2, \cdots, 10).\end{gathered} \qquad (2)$$

Every set of J's satisfying (1) and (2) reveals a space whose independent infinitesimal motions are represented by the infinitesimal operations of the set. Those functions of the elements that are invariant under these transformations will characterize the geometry of the space.

4. It may be verified without difficulty that the following forms for the fundamental transformations $J_1, J_2, \cdots, J_{10}$, which are functions of x, y, z, t, p, q, r, s, namely

$$p,\ q,\ r,\ s, \quad \begin{array}{lll} xq - yp, & yr - zq, & xs - tp, \\ zp - xr, & tr - zs, & ys - tq, \end{array} \qquad (3)$$

satisfy the conditions (1) and (2).

Let (x, y, z, t) and (x', y', z', t') be any two points, and (x_1, y_1, z_1, t_1), (x_1', y_1', z_1', t_1') be their positions after they are subjected to the transformation

$$I \equiv \sum_1^{10} a_i J_i,$$

where the J_i's have the value (3).

Let $\varphi(x, y, z, t, x', y', z', t')$ be a function which is absolutely invariant under this operation; then must

$$\varphi(x_1, y_1, z_1, t_1, x_1', y_1', z_1', t_1') = \varphi(x, y, z, t, x', y', z', t'),$$

that is

$$I\varphi \equiv 0$$

for all values of the a_i's; hence equating to zero the coefficients of the a_i's in $I\varphi$ we have the following system of linear partial differential equations for the function φ:

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial x'} = 0, \quad \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial y'} = 0, \quad \frac{\partial \varphi}{\partial z} + \frac{\partial \varphi}{\partial z'} = 0,$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t'} = 0,$$

$$x\frac{\partial \varphi}{\partial y} - y\frac{\partial \varphi}{\partial x} + x'\frac{\partial \varphi}{\partial y'} - y'\frac{\partial \varphi}{\partial x'} = 0, \quad y\frac{\partial \varphi}{\partial z} - z\frac{\partial \varphi}{\partial y} + y'\frac{\partial \varphi}{\partial z'} - z'\frac{\partial \varphi}{\partial y'} = 0,$$

$$x\frac{\partial \varphi}{\partial t} - t\frac{\partial \varphi}{\partial x} + x'\frac{\partial \varphi}{\partial t'} - t'\frac{\partial \varphi}{\partial x'} = 0,$$

$$z\frac{\partial \varphi}{\partial x} - x\frac{\partial \varphi}{\partial z} + z'\frac{\partial \varphi}{\partial x'} - x'\frac{\partial \varphi}{\partial z'} = 0, \quad t\frac{\partial \varphi}{\partial z} - z\frac{\partial \varphi}{\partial t} + t'\frac{\partial \varphi}{\partial z'} - z'\frac{\partial \varphi}{\partial t'} = 0,$$

$$y\frac{\partial \varphi}{\partial t} - t\frac{\partial \varphi}{\partial y} + y'\frac{\partial \varphi}{\partial t'} - t'\frac{\partial \varphi}{\partial y'} = 0.$$

This system of ten equations in eight variables should not in general possess a solution. However the equations are not all independent. If the first seven be multiplied respectively by the following functions

$$0, \quad 0, \quad \frac{xy' - x'y}{y - y'}, \quad 0, \quad \frac{z't - zt'}{yt' - y't}, \quad \frac{x' - x}{y - y'}, \quad \frac{y'z - yz'}{yt' - y't},$$

and the results be added, the eighth equation of the system is obtained; similarly, if the multipliers be

$$0,\quad 0,\quad \frac{yt'-y't}{y-y'},\quad \frac{x't-xt'}{x-x'},\quad 0,\quad \frac{t-t'}{y-y'},\quad \frac{z'-z}{x-x'},$$

the ninth equation results; and finally if the multipliers

$$0,\quad 0,\quad 0,\quad \frac{xy'-x'y}{x-x'},\quad \frac{zt'-z't}{xz'-x'z},\quad \frac{xt'-x't}{xz'-x'z},\quad \frac{y-y'}{x-x'},$$

be taken, the summation yields the tenth equation of the system.

The system composed of the first seven equations possesses at least one solution; that it has no more is readily seen by noting the fact that not all seventh order determinants of the matrix

$$\left\|\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ y'-y & x-x' & 0 & 0 & -y' & x' & 0 & 0 \\ 0 & z'-z & y-y' & 0 & 0 & -z' & y' & 0 \\ t'-t & 0 & 0 & x-x' & -t' & 0 & 0 & x' \end{array}\right\|$$

vanish; for example, the one formed by the last seven columns, whose value is written down at once as $(x-x')^2(y-y')$.

The unique solution of this system appears by observing that the first four equations demand that the solution be a function only of

$$X \equiv x-x',\quad Y \equiv y-y',\quad Z \equiv z-z',\quad T \equiv t-t';$$

the last three equations become

$$X\frac{\partial\varphi}{\partial Y} - Y\frac{\partial\varphi}{\partial X} = 0,\quad Y\frac{\partial\varphi}{\partial Z} - Z\frac{\partial\varphi}{\partial Y} = 0,\quad X\frac{\partial\varphi}{\partial T} - T\frac{\partial\varphi}{\partial X} = 0,$$

which require that φ be of the form

$$X^2 + Y^2 + Z^2 + T^2;$$

that is, the function

$$\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2 + (t-t')^2} \qquad (4)$$

is an absolute invariant under the most general transformaton of the group (3). This function is said to define the distance between the two points (x, y, z, t), (x', y', z', t').

5. Consider now the linear element manifoldness

$$\begin{aligned} x + lt + a &= 0, \\ y + mt + b &= 0, \\ z + nt + c &= 0. \end{aligned} \tag{5}$$

It is proposed to find the variations experienced by l, m, n, a, b, c, under the operations (3). This determination is readily effected in the following manner:

The total variations of (5) give

$$\delta(x + lt + a) = 0, \quad \delta(y + mt + b) = 0, \quad \delta(z + nt + c) = 0. \tag{6}$$

In (6) are substitued successively the increments assigned to x, y, z, t by the transformations (3). A comparison of the resulting equations with the equations (5) gives equations for the variations of l, m, n, a, b, c. The results are tabulated below:

Transformation.	δl	δm	δn	δa	δb	δc
p	0	0	0	-1	0	0
q	0	0	0	0	-1	0
r	0	0	0	0	0	-1
s	0	0	0	$-l$	$-m$	$-n$
$xq-yp$	$-m$	l	0	$-b$	a	0
$yr-zg$	0	$-n$	m	0	$-c$	b
$xs-tp$	l^2+1	lm	ln	al	am	an
$zp-xr$	n	0	$-l$	c	0	$-a$
$tr-zs$	$-ln$	$-mn$	$-(1+n^2)$	$-cl$	$-cm$	$-cn$
$ys-tq$	lm	$1+m^2$	mn	bl	bm	bn

Two linear manifoldnesses (l, m, n, a, b, c), (l', m', n', a', b', c') possess invariants under the preceding transformations. These appear as solutions of the simultaneous system composed of the ten equations

$$\frac{\partial\varphi}{\partial l}\delta l + \frac{\partial\varphi}{\partial m}\delta m + \frac{\partial\varphi}{\partial n}\delta n + \frac{\partial\varphi}{\partial a}\delta a + \frac{\partial\varphi}{\partial b}\delta b + \frac{\partial\varphi}{\delta c}\delta c + \frac{\partial\varphi}{\delta l'}\delta l' + \cdots = 0,$$

formed by assigning successively to the variations the values given in the above table. This system has two independent solutions. These are analogous in form to the expression for the distance between two straight lines in ordinary space. There exists however another unique invariant of the two linear manifoldnesses, namely

$$m\frac{\partial\varphi}{\partial l} - l\frac{\partial\varphi}{\partial m} + m'\frac{\partial\varphi}{\partial n'} - l'\frac{\partial\varphi}{\partial m'} = 0,$$

$$n\frac{\partial\varphi}{\partial m} - m\frac{\partial\varphi}{\partial n} + n'\frac{\partial\varphi}{\partial m'} - m'\frac{\partial\varphi}{\partial n'} = 0,$$

$$l\frac{\partial\varphi}{\partial n} - n\frac{\partial\varphi}{\partial l} + l'\frac{\partial\varphi}{\partial n'} - n'\frac{\partial\varphi}{\partial l'} = 0,$$

$$(l^2+1)\frac{\partial\varphi}{\partial l} + lm\frac{\partial\varphi}{\partial m} + ln\frac{\partial\varphi}{\partial n} + (l'^2+1)\frac{\partial\varphi}{\partial l'} + l'm'\frac{\partial\varphi}{\partial m'} + l'n'\frac{\partial\varphi}{\partial n'} = 0,$$

$$lm\frac{\partial\varphi}{\partial l} + (m^2+1)\frac{\partial\varphi}{\partial m} + nm\frac{\partial\varphi}{\partial n} + l'm'\frac{\partial\varphi}{\partial l'} + (m'^2+1)\frac{\partial\varphi}{\partial m'} + m'n'\frac{\partial\varphi}{\partial n'} = 0,$$

$$ln\frac{\partial\varphi}{\partial l} + mn\frac{\partial\varphi}{\partial m} + (n^2+1)\frac{\partial\varphi}{\partial n} + l'n'\frac{\partial\varphi}{\partial l'} + m'n\frac{\partial\varphi}{\partial m'} + (n'^2+1)\frac{\partial\varphi}{\partial n'} = 0,$$

which possesses a solution since the determinant of the coefficients is zero.

From the first three equations it is seen that φ is a function of

$$l^2+m^2+n^2, \quad l'^2+m'^2+n'^2, \quad ll'+mm'+nn';$$

calling these respectively u, v, and w, the last three equations become

$$2lu'\frac{\partial\varphi}{\partial u'} + 2l'v'\frac{\partial\varphi}{\partial v'} + (l+l')w'\frac{\partial\varphi}{\partial w'} = 0,$$

$$2mu'\frac{\partial\varphi}{\partial u'} + 2m'v'\frac{\partial\varphi}{\partial v'} + (m+m')u'\frac{\partial\varphi}{\partial w'} = 0,$$

$$2nu'\frac{\partial\varphi}{\partial u'} + 2n'v'\frac{\partial\varphi}{\partial v'} + (n+n')w'\frac{\partial\varphi}{\partial w'} = 0,$$

where

$$u' = u+1, \quad v' = v+1, \quad w' = w+1,$$

and, as should be the case, the determinant of the coefficients is zero. From any one of the three we have

$$\frac{du'}{u'}+\frac{dv'}{v'}=\frac{dw'}{w'};$$

that is, the solution φ is of the form $\dfrac{w'^2}{u'v'}$ or

$$\frac{(ll'+mm'+nn'+1)^2}{(l^2+m^2+n^2+1)(l'^2+m'^2+n'^2+1)}=\cos^2\theta, \qquad (7)$$

and the invariant θ is defined to be the angle between the two linear manifoldnesses.

The fundamental notions of distance and direction are thus introduced by the invariants (4) and (7), the former relative to two elements of the space, the latter relative to two simplest manifoldnesses composed of a simply infinite number of these elements. All the derived notions of geometry may then be developed by as simple extensions of these two primary notions from three dimensions to four dimensions as occur when passing from the plane to ordinary space.

6. A curve in a flat space of four dimensions is said to have triple curvature when not every five consecutive points of it lie in a euclidean space of three dimensions. Such a curve possesses not only rectilinear tangents, and osculating planes, but also osculating spaces of three dimensions; by the latter is meant the limiting position of a secant three-dimensional space Σ_3, $MM'M''M'''$ as the three points M', M'', M''' approach coincidence with M. At every point M of the curve we have to consider a tetrarectangular tetraeder which has as edges the tangent, the principal normal, the principal binormal, and the trinormal. By the principal normal is meant the only one of the ∞^2 normals at every point which lies in the osculating plane; by the principal binormal is meant the only one of the ∞^1 binormals at every point which lies in the osculating space; the trinormal is the only binormal perpendicular to the osculating space.

Taking the origin of the coördinate system at a point M movable along the curve, assume the tangent as x-axis, the trinormal as y-axis, the principal binormal as z-axis, and the principal normal as t-axis. Let $\delta\varphi$, $\delta\psi$, $\delta\chi$ be the respective angles between two consecutive tangents, two adjacent principal binormals, and two adjacent trinormals, when the origin is shifted from M to a neighboring point M' on the curve. The quadrangle of the direction cosines of the new axes referred to the old becomes

$$\begin{array}{c|cccc} & x & y & z & t \\ \hline x' & 1 & 0 & 0 & d\varphi \\ y' & 0 & 1 & d\chi & 0 \\ z' & 0 & -d\chi & 1 & d\psi \\ t' & -d\varphi & 0 & -d\psi & 1 \end{array} \qquad (11)$$

The limits of the ratios of $\delta\varphi$, $\delta\psi$, $\delta\chi$ to MM' as M' approaches M measure the curvatures of the curve at M; the first limit is called the flexion, the second the torsion, and the third the curvature; or perhaps more simply, the first, second, and third curvatures. Then if we put

$$ds = \rho d\varphi = \tau dx = r d\chi,$$

the numbers ρ, τ, r, the reciprocals of the curvatures, measure three lengths which are called the radii of the respective curvatures.

The array (11) shows that in order to discuss the curve in the domain of any of its points it is sufficient to know the functions φ, ψ, χ; that is, it is sufficient to give ρ, τ, r as functions of s. Three equations, the so-called intrinsic equations

$$f_1(\rho, \tau, r, s) = 0, \quad f_2(\rho, \tau, r, s) = 0, \quad f_3(\rho, \tau, r, s) = 0$$

which determine the radii of curvature as functions of s, determine at the same time the forms of those arcs of the curve which lie between points at which the tangent may become indeterminate; they do not fix the position of these branches in space.

7. The method of the intrinsic analysis employed so successfully by Professor E. Cesàro in his Lezioni di geometria intrinseca and developed by him for curves and surfaces in spaces of many dimensions may be exhibited for curves of triple curvature in detail in the following manner.

Let P be a point in space referred to M as origin and generally movable with M; let its coördinates be $x(s)$, $y(s)$ $z(s)$, $t(s)$. Let P' be the point of the trajectory of P corresponding to M'; its coördinates referred to the axes at M are $x + \delta x$, $y + \delta y$, $z + \delta z$, $t + \delta t$; its coördinates referred to the axes at M' are $x + dx$, $y + dy$, $z + dz$, $t + dt$; finally let u, v, w, p be the coördinates of M' referred to M. Then projecting $M'P'$ on the axes at M, we have by virtue of the array (11):

$$\begin{aligned} x + \delta x &= u + x + d\chi - (t + dt)\, d\varphi, \\ y + \delta y &= v + y + dy - (z + dz)\, d\chi, \\ z + \delta z &= w + z + dz + (y + dy)\, d\chi - (t + dt)\, d\psi, \\ t + \delta t &= p + t + dt + (x + da)\, d\varphi + (z + dz)\, d\psi. \end{aligned} \qquad (12)$$

Excluding from further consideration here those curves for which it is not legitimate to assume that the limit of the ratio of the arc to the arc is unity, we have from the definition of the tangent as the limiting position of a secant

$$\pounds\, \frac{u}{\delta s} = 1, \qquad \pounds\, \frac{v}{\delta s} = \pounds\, \frac{w}{\delta s} = \pounds\, \frac{p}{\delta s} = 0,$$

and the formulæ (12) give Cesàro's fundamental relations

$$\begin{aligned} \frac{\delta x}{ds} &= \frac{dx}{ds} - \frac{t}{\rho} + 1, & \frac{\delta y}{ds} &= \frac{dy}{ds} - \frac{z}{r}, \\ \frac{\delta z}{ds} &= \frac{dz}{ds} + \frac{y}{r} - \frac{t}{\tau}, & \frac{\delta t}{ds} &= \frac{dt}{ds} + \frac{x}{\rho} + \frac{z}{\tau}. \end{aligned} \qquad (13)$$

The formulæ (12) are also applicable to the cosines α, β, γ, ε that define any direction, provided that u, v, w, p are zero, and the corresponding fundamental formulæ for this direction are found by replacing x, y, z, t in (13) respectively by α, β, γ, ε.

From (13) the necessary and sufficient conditions for the immobility of a point are

$$\frac{dx}{ds} = \frac{t}{\rho} - 1, \quad \frac{dy}{ds} = \frac{z}{r}, \quad \frac{dz}{ds} = \frac{t}{\tau} - \frac{y}{r}, \quad \frac{dt}{ds} = -\frac{x}{\delta} - \frac{\gamma}{\tau}. \qquad (14)$$

8. The above fundamental formulæ may be employed to show that every trio of intrinsic equations determines a curve uniquely, at least between convenient limits of s in a range where the curvatures are finite and continuous functions of the arc. In fact there exists, by the theory of systems of linear differential equations, one and but one quatrain of functions $x(s)$, $y(s)$, $z(s)$, $t(s)$ satisfying the simultaneous system (14), and reducing to a, b, c, e for $s = 0$. These functions x, y, z, t are evidently the coördinates with regard to the tetraeder of origin M of that point whose coördinates with regard to the tetraeder of origin O are a, b, c, e.

As remarked above, the necessary and sufficient conditions for the invariability of the direction $(\alpha, \beta, \gamma, \varepsilon)$ are

$$\frac{dx}{ds}=\frac{\varepsilon}{\rho}, \quad \frac{d\beta}{ds}=\frac{\gamma}{r}, \quad \frac{d\gamma}{ds}=-\frac{\beta}{r}+\frac{\varepsilon}{\tau}=, \quad \frac{d\varepsilon}{dt}=-\frac{\alpha}{\rho}-\frac{\gamma}{\tau}; \tag{15}$$

if $(\alpha, \beta, \gamma, \varepsilon,)$ and $(\alpha', \beta', \gamma', \varepsilon')$ are any two solutions of this system it follows from the equations of the system that

$$\frac{d}{ds}(\alpha\alpha' + \beta\beta' + \gamma\gamma' + \varepsilon\varepsilon') = 0, \quad \frac{d}{ds}(\alpha^2 + \beta^2 + \gamma^2 + \varepsilon^2) = 0,$$

$$\frac{d}{ds}(\alpha'^2 + \beta'^2 + \gamma'^2 + \varepsilon'^2) = 0;$$

that is

$$S\alpha\alpha', \qquad S\alpha^2, \qquad S\alpha'^2$$

are integrals of (15).

Hence if there be determined four systems of functions contained in the first of the following arrays

$$\begin{array}{cccc} \alpha_1 & \beta_1 & \gamma_1 & \varepsilon_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \varepsilon_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \varepsilon_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \varepsilon_4 \end{array}, \quad \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}$$

in such a manner that they satisfy (15) and take the corresponding values in the second array for $s = 0$, we have

$$S\alpha_i\alpha_j = 0, \quad (i \neq j); \quad S\alpha_i\alpha_j = 1, \quad (i = j);$$

that is, the determinant represented by the first array is orthogonal. The elements of this determinant are the direction cosines of axes of origin O with respect to those of origin M, since they give the directions of the former for $s = 0$, and satisfy the conditions (15) for the invariability of those directions. Then in all cases

$$x = x_0 + a\alpha_1 + b\alpha_2 + c\alpha_3 + e\alpha_4, \qquad y = y_0 + a\beta_1 + b\beta_2 + c\beta_3 + e\beta_4,$$

$$z = z_0 + a\gamma_1 + b\gamma_2 + c\gamma_3 + e\gamma_4, \qquad t = t_0 + a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 + e\varepsilon_4;$$

for each value of s the constants a, b, c, e, can be determined in such manner that they represent the coördinates of M with respect to axes of origin O; for this purpose it is sufficient to put x, y, z, t equal to zero in the preceding equations and solve for a, b, c, e:

$$\begin{aligned} a &= -(\alpha_1 x_0 + \beta_1 y_0 + \gamma_1 z_0 + \varepsilon_1 t_0), \quad b = -(\alpha_2 x_0 + \beta_2 y_0 + \gamma_2 z_0 + \varepsilon_2 t_0), \\ c &= -(\alpha_3 x_0 + \beta_3 y_0 + \gamma_3 z_0 + \varepsilon_3 t_0), \quad e = -(\alpha_4 x_0 + \beta_4 y_0 + \gamma_4 z_0 + \varepsilon_4 t_0). \end{aligned} \tag{16}$$

Thus the curve defined by a trio of intrinsic equations is uniquely determined; by varying s the coördinates (a, b, c, e) of all of its points with regard to a system of immovable axes are known.

9. It may be observed in conclusion that the whole theory of curves of triple curvature may be deduced by differentiation and elimination by means of the fundamental invariance conditions (14) and (15).

Differentiating (16) we have

$$\frac{da}{ds} = -S\left(x_0\frac{dx_0}{ds} + a_1\frac{dx_0}{ds}\right) = a_1, \quad \frac{db}{ds} = a_2, \quad \frac{dc}{ds} = a_3, \quad \frac{de}{ds} = a_4;$$

also

$$\frac{d^2a}{ds^2} = \frac{\varepsilon_1}{\rho}, \cdots;$$

$$\frac{d^3a}{ds^3} = \varepsilon_1\frac{d}{ds}\frac{1}{\rho} - \frac{a_1}{\rho^2} - \frac{\gamma_1}{\rho\tau}, \cdots.$$

These formulæ lead to the following expressions for the curvatures as functions of the arc s: for the first curvature, or flexion, which measures the rapidity with which the curve deviates from being a straight line,

$$\frac{1}{\rho^2} = \left(\frac{d^2a}{ds^2}\right)^2 + \left(\frac{d^2b}{ds^2}\right)^2 + \left(\frac{d^2c}{ds^2}\right)^2 + \left(\frac{d^2e}{ds^2}\right)^2;$$

for the second curvature, or torsion, which expresses the rapidity with which the curve deviates from being a plane curve,

$$\frac{1}{\tau^2} = \frac{S\left(\frac{d^2a}{ds^2}\right)^2 S\left(\frac{d^3a}{ds^3}\right)^2 - \left\{S\left(\frac{d^2a}{ds^2}\right)^2\right\}^3 - \left\{S\frac{d^2a}{ds^2}\frac{d^3a}{ds^3}\right\}}{\left\{S\left(\frac{d^2a}{ds^2}\right)^2\right\}^2};$$

finally for the third curvature, which measures the rapidity of deviation from a curve of double curvature,

$$\frac{1}{r^2} = \left(\frac{dA}{ds}\right)^2 + \left(\frac{dB}{ds}\right)^2 + \left(\frac{dC}{ds}\right)^2 + \left(\frac{dE}{ds}\right)^2,$$

where $A = A'/D,\ B = B'/D,\ C = C'/D,\ E = E'/D,$
$D^2 = A'^2 + B'^2 + C'^2 + E'^2,$

A', B', C', E' being the minors corresponding to a, b, c, e in the Wronskian

$$W = \left| a, \frac{db}{ds}, \frac{d^2c}{ds^2}, \frac{d^3e}{ds^3} \right|.$$

The above values for the curvatures are derived directly by Pirondini in *Battaglini's Journal* for 1890.

PRINCETON, NEW JERSEY.

NOTES.

THE closing (October) number of Volume I of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On surfaces enveloped by spheres belonging to a linear spherical complex," by P. F. SMITH; "On certain relations among the theta constants," by J. I. HUTCHINSON; "On groups which have the same group of isomorphisms," by G. A. MILLER; "Die Hesse'sche und die Cayley'sche Curve," by P. GORDAN; "Application of a method of D'Alembert to the proof of Sturm's theorem of comparison," by M. BÔCHER; "Two plane movements generating quartic scrolls," by E. M. BLAKE; "The invariant theory of the inversion group: geometry upon a quadric surface," by E. KASNER; "A simple proof of the fundamental Cauchy-Goursat theorem." by E. H. MOORE; "Notes and errata: Volume I."

THE mathematical section of the British association for the advancement of science held its meeting at Bradford, September 10, 1900, with Major P. A. MacMahon, F.R.S. as presiding officer, and Mr. E. T. Whittaker, secretary. The following papers were read:

(1) MISS F. HARDCASTLE: Report (preliminary) on the present state of the theory of point groups.

(2) MAJOR P. A. MACMAHON: "A property of the characteristic symbolic determinant of any n quantics in n variables."

(3) PROFESSOR CYPARISSOS STEPHANOS: "Sur les relations entre la géométrie projective et la mécanique."

(4) MR. H. S. CARSLAW: "The use of multiple space in applied mathematics."

(5) MAJOR P. A. MACMAHON: "The asyzygetic and perpetuant covariants of systems of binary quantics."

(6) Lieut.-Col. Allan Cunningham: "Determination of successive high primes."

(7) Mr. A. B. Basset: "A quintic curve cannot have more than fifteen real points of inflexion."

(8) Major P. A. MacMahon: "On the symbolism appropriate to the study of orthogonal and Boolian invariant systems which appertain to binary and other quantics."

(9) Dr. J. Willis: "On the construction of magic squares."

(10) Professor J. D. Everett: "On a central-difference interpolation formula."

(11) Professor Everett: "On Newton's contributions to central-difference interpolation."

The committee appointed in 1888 to calculate tables of certain mathematical functions presented a report of their year's progress. The work on which they have for some time been engaged, namely, the preparation of a new "Canon Arithmeticus," is now almost completed. The calculations were made by Lieut.-Col. Allan Cunningham.

For volumes IV. and V. of the Encyclopædia of the mathematical sciences, edited by H. Burkhardt, and W. Fr. Meyer (see Bulletin, 2d series, volume 3, p. 326; volume 4, p. 32; volume 5, pp. 109, 151, 202, 266); the following list of collaborators and subjects is announced:

Volume IV: Mechanics.—*A*. Introduction: 1. The principles of rational mechanics, A. Voss, of Würzburg.—*B*. Mechanics of Points and Rigid Systems: I. Geometric foundation and development of concrete modes of representation: 2. Theory of systems of vectors and screws, H. E. Timerding, of Strassburg; 3, Geometry of masses, G. Jung, of Milan; 4. Kinematics, A. Schoenflies, of Königsberg; 5. Elementary statics, including graphical statics, L. Henneberg, of Darmstadt; 6. Elementary kinetics, J. Petersen, of Copenhagen. II. General analytical treatment of arbitrary systems with finite degrees of freedom: 7. General developments, P. Stäckel, of Kiel; 8. Complete treatment of special cases, P. Stäckel of Kiel; 9. Mathematical treatment of the problem of n bodies, E. T. Whittaker, of Cambridge, Eng. III. Applications, with consideration of disturbing influences: 10. Mechanics of the simplest kinds of physical apparatus and of experiments, Ph. Furtwängler, of Potsdam; 11. Kinetic problems of machines, K. Heun, of Berlin; 12. Ballistics, C. Cranz, of Stuttgart; 13. Sports, G. T. Walker, of Cambridge, Eng.—*C*. Mechanics of Deformable Bodies. I. Analytical geometrical aids: 14. Vector analysis, M. ABRAHAM, of Göttingen. II. Hydrodynamics: 15. Physical foundation, A. E. H. LOVE, of Oxford; 16. Theoretical developments, A. E. H. LOVE, of Oxford; 17. Hydraulics, part one: Flow of water in pipes, canals, etc., E. PALADINI, of Milan; 18. Hydraulics, part two: Motors and pumps, M. GRÜBLER, of Charlottenburg; 19. Motion of ships, A. KRILOFF, of St. Petersburg; 20. Aerodynamics, S. FINSTERWALDER, of Munich. III. Theory of elasticity: 21. Physical foundation, A. SOMMERFELD, of Aachen; 22. Theoretical treatment of statical problems, O. TEDONE, of Genoa; 23. Statics of building construction, E. OVAZZA, of Palermo; 24. Vibrations, in particular acoustics, H. LAMB, of Manchester, Eng.; 25. Theory of measuring apparatus based upon elastic effects, PH. FURTWÄNGLER, of Potsdam.—*D*. Mechanics of Systems Consisting of very Numerous Discrete Particles: 26. The application of the theory of probabilities, L. BOLTZMANN, of Leipsic.

Volume V: Physics.—*A*. Introduction: 1. Measures and measurement, C. RUNGE, of Hanover; 2. General properties of bodies, Gravity, J. ZENNECK, of Strassburg.—*B*. Thermodynamics: 3. General foundations of thermodynamics, G. H. BRYAN, of Upper Bangor, Wales; 4. Special substances and conditions, H. KAMERLINGH-ONNES, of Leiden, and J. KORTEWEG, of Amsterdam; 5. Dissipation of energy, including conduction of heat, E. W. HOBSON, of Cambridge, Eng.; 6. Technical theory of heat, H. LORENZ, of Halle, and E. RIECKE, of Göttingen.—*C*. Molecular Physics: 7. Fundamental notions concerning atoms and molecules; *a*. Introductory remarks on the atomic theory, L. BOLTZMANN, of Leipsic; *b*. The mathematical foundations of chemistry, W. MEYERHOFFER, of Berlin; 8. Crystallography; *a*. Theory of symmetry and structure, A. SCHOENFLIESS and O. MÜGGE, of Königsberg; *b*. Computation and drawing of crystals, TH. LIEBISCH, of Göttingen; 9. Kinetic theory of gases, L. BOLTZMANN, of Leipsic; 10. Capillarity and cohesion, H. MINKOWSKI, of Zurich; 11. Physical chemistry and electrochemistry, J. H. VAN'T HOFF, of Berlin.—*D*. Electricity and Optics: Physical foundations of the theory of electricity. 12. Action at a distance, laws governing elements, R. REIFF, of Stuttgart; 13. Action of a field, Maxwell's theory and allied topics, H. A. LORENTZ, of Leiden; 14. Further developments of Maxwell's theory, Electron-theory, H. A. LORENTZ, of Leiden.—Special mathematical developments in the theory of electricity. 15. Electrostatics and magnetostatics, H. M.

MACDONALD, of Cambridge, Eng.; 16. Relations between electricity and elastic deformation, F. POCKELS, of Heidelberg; 17. Stationary and slowly changing fields (electric currents, induction, and electrodynamics in the narrower sense), A. TAUBER, of Vienna; 18. Relations of the electric current to heat and magnetism, H. DIESSELHORST, of Berlin; 19. Rapidly changing fields, M. ABRAHAM, of Göttingen; 20. Technical electricity, TH. DESCOUDRES, of Göttingen.—Physical foundations of optics. 21. Older theories, A. WANGERIN, of Halle; 22. Electromagnetic theory of light, W. WIEN, of Würzburg; 23. The part played by molecular physics and the electron-theory in optics, W. WIEN, of Würzburg.—Special mathematical developments in optics. 24. Ray treatment of optics and optical instruments, S. FINSTERWALDER, of Munich; 25. Wave treatment of optics (interference and refraction), K. STREHL, of Erlangen; 26. Optics of crystals, F. POCKELS, of Heidelberg.—*E.* Conclusion: 27. General physical notions and methods, A. SOMMERFELD, of Aachen, and J. LARMOR, of Cambridge, Eng.

CAMBRIDGE UNIVERSITY.—Advanced mathematical courses for the current academic year are announced as follows:

Michaelmas term. 1900:—By Professor Sir G. G. STOKES: Hydrodynamics, three hours.—By Professor A. R. FORSYTH: Abel's theorem and abelian functions, three hours; Invariants and covariants, two hours.—By Professor G. H. DARWIN: Lunar theory, three hours.—By Mr. R. PENDLEBURY: Theory of equations, three hours.—By Dr. E. W. HOBSON: Spherical and cylindrical harmonics, three hours.—By Mr. J. LARMOR: Electricity and magnetism, three hours.—By Mr. H. F. BAKER: Continuous groups, three hours.—By. Mr. H. M. MACDONALD: Waves (especially waves of light), three hours.—By Mr. H. W. RICHMOND: Plane analytical geometry, three hours.—By Mr. G. T. WALKER: The electro-magnetic field, three hours.—By Mr. E. T. WHITTAKER: The problem of three bodies, three hours.—By Mr. J. H. GRACE: Invariants and geometrical applications, three hours.

Lent term, 1901:—By Professor Sir G. G. STOKES: Physical optics, three hours.—By Professor A. R. FORSYTH: Invariants and covariants (symbolical methods), three hours.—Fourier's and other expansion theorems, two hours.—By Professor Sir R. S. BALL: Planetary theory, three hours.—By Mr. R. PENDLEBURY: Theory of numbers, three hours.—By Dr. E. W. HOBSON: Vibrations and sound, three hours.—By Mr. J. LARMOR : Electricity and magnetism (continued, more advanced), three hours.—By Mr. H. F. BAKER : Discontinuous groups, three hours.—By Mr. H. M. MACDONALD : Elasticity, three hours.—By Dr. J. W. L. GLAISHER : Elliptic functions, three hours.—By Mr. A. BERRY : Elliptic functions, three hours.—By Mr. G. T. WALKER : Physical optics, three hours.—By Mr. G. T. BENNETT : Linear and quadratic complexes, three hours.—By Mr. E. T. WHITTAKER : Linear differential equations of the second order.

Easter term, 1901 :—By Professor G. H. DARWIN : Figure of the earth and precession, three hours.—By Mr. W. L. MOLLISON : Theory of potential and electrostatics.—By Mr. A. N. WHITEHEAD : Non-euclidean geometry.—By Mr. G. T. WALKER : Rigid dynamics, three hours.—By Mr. I'A. BROMWICH : Algebra of matrices and bilinear forms.

Long vacation, 1901 :—By Professor Sir R. S. BALL : Theory of screws, two hours.

THE several universities below offer during the winter semester 1900–1901 courses in mathematics as follows :

UNIVERSITY OF STRASSBURG.—By Professor T. REYE : Geometry of position, three hours ; Analytical mechanics, two hours ; Seminar, two hours.—By Professor H. WEBER : Elliptic functions, four hours ; Selected chapters of higher algebra, two hours ; Seminar in coöperation with Dr. WELLSTEIN.—By Professor G. ROTH : Algebraic analysis and determinants, three hours ; Analytical geometry of space, two hours ; Ordinary differential equations, two hours.—By Professor A. KRAZER : Infinitesimal calculus, four hours ; Plane analytical geometry, three hours ; Seminar, two hours.—By Dr. H. E. TIMERDING : Introduction to higher analysis, two hours ; Elasticity and rigidity, one hour.—By Dr. J. WELLSTEIN : Descriptive geometry I. two hours, with exercises ; Seminar, with Professor WEBER.

UNIVERSITY OF TÜBINGEN.—By Professor A. v. BRILL : Introduction to higher mathematics, four hours ; On non-rigid systems and the mechanics of Hertz, three hours ; Seminar, two hours.—By Professor H. STAHL : Higher algebra, three hours ; application of the theory of functions, four hours ; Seminar, two hours.—By Professor L. MAURER : Higher analysis II, two hours, with exercises, one hour ; Spherical trigonometry, one hour, with exercises, one hour ; Theory of numbers, two hours.

UNIVERSITY OF WÜRZBURG.—By Professor F. PRYM :

Differential calculus, four hours, with exercises, two hours; Higher theory of functions, four hours; Seminar in higher mathematics, two hours.—By Professor A. Voss: Introduction to the theory of differential equations, four hours; Application of the infinitesimal calculus to the theory of curves and surfaces, four hours; Seminar, in analytical and synthetical geometry of conics, two hours.

UNIVERSITY OF VIENNA.—By Professor J. v. ESCHERICH: Definite integrals and differential equations, five hours; Proseminar, one hour; Seminar, two hours.—By Professor L. GEGENBAUER: Algebra III, two hours; Theory of numbers, three hours; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: Elements of infinitesimal calculus for students of all faculties, with exercises, five hours; Proseminar, one hour; Seminar, two hours.—By Professor G. KOHN: Introduction to synthetic geometry, four hours, with exercises, one hour; Algebraic plane curves, one hour.—By Dr. V. SERSAWY: Mathematics of insurance, two courses of three and four hours, respectively.—By Dr. A. TAUBER: Analytical mechanics, three hours; Mathematics of insurance, four hours, with exercises, two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics, three hours.—By Dr. K. ZSIGMONDY: Surfaces of second order, one hour.—By Dr. R. D. v. STERNECK: Differential geometry, three hours.

UNIVERSITY OF ZÜRICH.—By Professor H. BURKHARDT: Infinitesimal calculus, four hours; Differential equations, four hours; Seminar, two hours.—By Professor A. WEILER: Analytical geometry, with exercises, four hours; Descriptive geometry, with exercises, four hours; Synthetic geometry, two hours; Mathematical geography, two hours.—By Dr. F. KRAFT: Modern synthetic geometry, four hours; Analytical geometry, two hours; Analytical mechanics, four hours; Exercises in higher mathematics, two hours.—By Dr. E. GUBLER: Theory and resolution of higher equations, with exercises, three hours; Methods of mathematical instruction in secondary schools, two hours; Spherical trigonometry, one hour.

THE Göttingen Academy of Sciences has awarded 800 marks to Professor F. KLEIN for his work on the Mathematical Encyclopædia.

PROFESSOR W. DYCK has been appointed director of the Technical High School in Munich for the next three years.

DR. J. B. FAUGHT, recently assistant professor of mathematics in the University of Indiana, has been appointed professor of mathematics in the Northern Michigan State Normal School, Marquette, Mich.

DR. D. F. CAMPBELL, of the Lawrence Scientific School, Harvard University, has been appointed instructor in mathematics in the Armour Institute of Technology, Chicago, Ill.

DR. G. A. BLISS has been appointed instructor in mathematics at the University of Minnesota.

MR. W. B. FORD has been appointed instructor in mathematics in the University of Michigan.

PROFESSOR F. H. LOUD has resumed his duties at Colorado College after two years' leave of absence.

PROFESSOR E. J. TOWNSEND has returned to his position at the University of Illinois after two years of study at the University of Göttingen.

PROFESSOR I. STRINGHAM has returned to the University of California, after a year of travel in Europe.

PROFESSOR B. O. PEIRCE, of Harvard University, has been granted a year's leave of absence.

PROFESSOR S. J. BROWN, astronomical director of the Naval Observatory, has been appointed superintendent of the Nautical Almanac, to succeed Professor H. T. TODD, who was retired by age on August 25th.

PROFESSOR R. E. ALLARDICE, of Leland Stanford University, is spending a year abroad on leave of absence.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABEL (N. H.). Abhandlung über eine besondere Klasse algebraisch auflösbarer Gleichungen. (1829.) Herausgegeben von A. Loewy. Leipzig, Engelmann, 1900. 12mo. 50 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 111.) Cloth. M. 0.90

BEER (F.). Kriterien für die Irrazionalität von Funktionalwerten. (Diss.) Göttingen, 1899. 8vo. 61 pp.

BEINHORN (J.). Zur Theorie der quadratischen Formen. (Diss.) Marburg, 1900. 8vo. 42 pp.

BINDER (J. K.). Ueber eine gewisse Abbildung zweier Rotationshyperboloide auf einander. (Diss.) Leipzig, 1900. 8vo. 81 pp., 4 plates.

BOEHM (K.). Zur Integration partieller Differentialsysteme. (Habilitationsschrift.) Leipzig, Teubner, 1900. 8vo. 55 pp. M. 1.80

BORNEMANN (G.). Analytische Studien. (Progr.) Chemnitz, 1899. 4to. 28 pp.

BURONI (S.). See LAVAGGI.

CAUCHY (A. L.). Abhandlung über bestimmte Integrale zwischen imaginären Grenzen. (1825.) Herausgegeben von P. Stäckel. Leipzig, Engelmann, 1900. 12mo. 80 pp (Ostwald's Klassiker der exakten Wissenschaften, No. 112.) Cloth. M. 1 25

———. See LAGRANGE.

ENGEL (F.). Sophus Lie. Ausführliches Verzeichnis seiner Schriften. Mit dem Bildnis Sophus Lies in Heliogravüre. Leipzig, Teubner, 1900. 8vo. 42 pp. M. 2.00

FABBRI (E.). Il teorema dell'integrale di Cauchy; contributo alla storia critica dell'analisi. Bologna, Zamorani e Albertazzi, 1900. 8vo 71 pp.

FLECHSENHAAR (A.). Ueber Multiplizität von Gleichungen. (Diss.) Giessen, 1899. 8vo. 27 pp.

GUTZMER (A.), HAUCK (G.), HENSEL (K.), HEUN (K.). See JAHRESBERICHT.

JAHRESBERICHT der Deutschen Mathematiker-Vereinigung. Vol. 8, Heft 2, enthaltend: Die Entwickelung der Lehre von den Punktmannigfaltigkeiten. Bericht, erstattet der Deutschen Mathematiker-Vereinigung von A. Schoenflies. Herausgegeben im Auftrage des Vorstandes von G. Hauck und A. Gutzmer. Leipzig, Teubner, 1900. 8vo. 4 + 6 + 251 pp.

———. Vol. 9, Heft 2, enthaltend: Die kinetischen Probleme der wissenschaftlichen Technik. Bericht, erstattet der Deutschen Mathematiker-Vereinigung von K. Heun. Herausgegeben im Auftrage des Vorstandes von K. Hensel und A. Gutzmer. Leipzig, Teubner, 1900. 8vo. 6 + 123 pp.

KLEIBER (J.). Ueber die Priminvarianten quadratischer Formen beliebig hoher Stufe. (Progr.) München, 1900. 8vo. 58 pp.

KOWALEWSKI (G.). See LAGRANGE.

LAGRANGE und CAUCHY. Zwei Abhandlungen zur Theorie der partiellen Differenzialgleichungen erster Ordnung. (1772 und 1819.) Herausgegeben von G. Kowalewski. Leipzig, Engelmann, 1900. 12mo. 54 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 113.) Cloth. M. 1.00

LAVAGGI. Calcolo infinitesimale; lezioni [dettate nell'anno] 1899–1900 nella r. università di Parma, compilate per cura di S. Buroni. Parma, Zafferri, 1900. 8vo. Pp. 433–560.

LEITZMANN (H.). See PASCAL (E.).

LÉVY (L.). See ROUCHÉ (E.).

LOEWY (A.). See ABEL (N. H.).

MEINEL (K.). Ueber Potenzlinien und Potenzkreise, das Apollonische Taktionsproblem und die Malfattische Aufgabe. (Progr.) Furth, 1900. 8vo. 33 pp., 3 plates.

PASCAL (E.). Repertorium der höheren Mathematik (Definitionen, Formeln, Theoreme, Literatur). Autorisirte deutsche Ausgabe nach einer neuen Bearbeitung des Originals von A. Schepp. In 2 Teilen: Analysis und Geometrie. Teil I: Analysis. Leipzig, Teubner, 1900. 8vo. 12 + 638 pp. Cloth. M. 10.00

——. Die Determinanten. Eine Darstellung ihrer Theorie und Anwendungen mit Rucksicht auf die Gesamtheit der neueren Forschungen. Deutsch von H. Leitzmann. Leipzig, Teubner, 1900. 8vo. 16 + 266 pp. M. 10.00

ROUCHÉ (E.) et LÉVY (L.). Analyse infinitésimale, à l'usage des ingénieurs. Vol. 1: Calcul différentiel. Paris, Gauthier-Villars, 1900. 8vo. 8 + 559 pp. (Encyclopédie industrielle.)

SCHEPP (A.). See PASCAL (E.).

SCHERRER (O.) Ueber Kegelschnitte im Raume. (Progr.) Frauenfeld, 1900. 4to. 35 pp.

SCHILLING (F.). Ueber die Nomographie von M. d'Ocagne. Eine Einführung in dieses Gebiet. Leipzig, Teubner, 1900. 8vo. 47 pp. M. 2.00

SCHMIDT (R.). Beiträge zum Gesetze der kleinen Zahlen. (Diss.) Göttingen, 1900. 8vo. 56 pp.

SCHOENFLIES (A.). See JAHRESBERICHT.

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LIBRARY.

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 27, 1900, extending through the usual morning and afternoon sessions. Thirty-two persons were in attendance, including the following twenty-five members of the Society:

Professor E. W. Brown, Professor F. N. Cole, Dr. J. W. Davis, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Miss Ida Griffiths, Miss Carrie Hammerslough, Dr. A. A. Himowich, Professor Harold Jacoby, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Dr. G. A. Miller, Professor R. W. Prentiss, Dr. P. L. Saurel, Professor P. F. Smith, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Professor R. S. Woodward.

Professor Thomas S. Fiske, Vice-President of the Society, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor G. L. Brown, South Dakota Agricultural College, Brookings, So. Dak.; Mr. C. H. Davis, New York, N. Y.; Dr. D. N. Lehmer, University of California, Berkeley, Cal.; Miss Ida M. Schottenfels, Chicago, Ill.; Professor F. D. Sherman, Columbia University, New York, N. Y.; Mr. Burke Smith, Northwestern University, Evanston, Ill. Twenty-five applications for membership were received.

It was decided to hold the next summer meeting and colloquium of the Society at Cornell University. A committee, consisting of Professors W. F. Osgood, James Pierpont, J. H. Tanner, and H. S. White, was appointed to coöperate with the Secretary in making the necessary arrangements. The conditions are favorable for extending the colloquium through a period of two or more weeks.

The following papers were read at this meeting:

(1) Professor MAXIME BÔCHER: "On linear dependence of functions of one variable."

(2) Professor DAVID HILBERT: "Ueber Flächen von constanter Gauss'cher Krümmung."

(3) Professor E. O. LOVETT: "Three notes on the geometry of contact transformations."

(4) Dr. G. A. MILLER: "On a theorem in substitutions."

(5) Mr. H. W. KUHN: "Several theorems on imprimitive groups."

(6) Professor S. L. PENFIELD ; "The solution of spherical triangles by graphical methods, with exhibition of scales and protractors."

(7) Miss I. M. SCHOTTENFELS : "On a set of definitional functional properties for the analytical function

$$f(z) = \frac{\tan \pi z}{\pi}."$$

(8) Professor P. F. SMITH : "Geometry within a linear spherical complex."

(9) Dr. E. J. WILCZYNSKI : "Invariants of systems of linear differential equations."

Professor Hilbert's paper was presented to the Society through Professor E. H. Moore, Mr. Kuhn's through Dr. G. A. Miller, and Miss Schottenfels's through the Secretary. Professor Penfield was introduced by Professor P. F. Smith. In the absence of the authors, Mr. Kuhn's paper was read by Dr. Miller, and the papers of Professor Bôcher, Professor Hilbert, Professor Lovett, Miss Schottenfels, and Dr. Wilczynski were read by title.

Professor Bôcher's paper appears in the present number of the BULLETIN. Abstracts of the remaining papers are given below.

Professor Hilbert's paper will appear in an early number of the *Transactions*. Following Beltrami, it is known that a piece free of singularities of a surface of negative constant curvature makes real in the domain of ordinary euclidean space a piece of a Lobachevsky (non-euclidean) plane, the straight lines of the Lobachevsky plane corresponding to the geodetic lines of the surface of negative constant curvature and the distances and angles in the plane being the real distances and angles on the surface. Of the known surfaces of negative constant curvature there is none which in all its extent is continuous and has a continuously varying tangent plane ; on the contrary all such known surfaces have singular lines beyond which a continuation of the surface with preservation of the properties just mentioned is impossible. The first part of this memoir concludes with the theorem that there exists no surface of negative constant curvature which in all its extent is continuous and has a continuously varying tangent plane, and hence that the complete Lobachevsky plane cannot by the method of Beltrami be realized in ordinary euclidean space.

The principal theorem of the second part is this: On any simlpy or multiply connected closed region entirely in the finite and free of singularities lying on an analytic surface of positive constant curvature + 1 the greater of the two principal radii of curvature of the surface at the various points of the region takes its maximum value at a point of a boundary, and at an interior point only if the surface is a part of the sphere of radius 1. From this follows immediately a new proof of the theorem proved by Liebmann (*Göttinger Nachrichten*, 1899) that the sphere of radius 1 is the only closed analytic surface free of singularities and of positive constant curvature +1.

Professor Lovett's paper consists of three notes. The first is a collection of original problems relative to the contact transformations between the principle elements of ordinary space, namely points, planes, straight lines, and spheres, and has been sent to the *Annals of Mathematics* for publication.

The second constructs the invariants of an n dimensional space as was done for four dimensional space in a recent number of the BULLETIN. It has been offered to the *American Journal of Mathematics*.

The third determines all contact transformations between singular manifoldnesses of surface elements and employs the results to effect the integration of the equations defining certain other categories of transformations. A surface element is the ensemble of a point and a plane through it, the dimensions of the plane being infinitesimal. Two surface elements are said to be associated when the point of each lies in the plane of the other to terms of the first order. An element manifoldness of surface elements is an aggregate such that every element of the aggregate is associated with its infinitely near neighboring element. These notions of surface element and element manifoldness were introduced by Lie. Now among these element manifoldnesses there exist certain ones which enjoy the property that every element of the aggregate is associated not only with its nearest neighbor but also with every other element of the family. Such element manifoldnesses might be called singular. There are but three varieties of such manifoldnesses, namely, the aggregates of all the surface elements of a point, of all those appertaining to a straight line, and of all the elements belonging to a plane. If it be proposed then to determine all the contact transformations which change singular element manifoldnesses into such, the following cases call for further examination: 1° those which change point into point; 2° those which change point into point and plane into plane; 3° those which change point into plane and plane into point; 4° those which change straight line into straight line; 5° those which change plane into plane. The first three categories are well known to consist respectively of the infinite group of all extended point transformations, of the finite group of all projective transformations, and of all dualistic transformations. The fourth category is equally well known to consist of the second and third categories, a result which yields itself immediately from the above definition of singular element manifoldness. The fifth category which does not seem to have been constructed is readily designed geometrically from the products DPD, where D is the general dualistic transformation and P is an arbitrary point transformation.

Dr. Miller's paper is in abstract as follows: Let s_1, s_2 be any two substitutions which involve the same n letters, and let G_1, G_2 be the groups which are composed of all the substitutions in these n letters that are commutative with s_1, s_2 respectively. The necessary and sufficient condition that G_1, G_2 represent the same substitution group is that s_1, s_2 are similar. All the substitutions of degree m that are commutative with s_1 form a group. When $m = n + b$ and this group involves all the letters of s_1, it is the direct product of G_1 and the symmetric group of degree b. Two dissimilar substitutions of degree n give rise to two distinct groups of degree m. Hence every substitution s_1 gives rise to an infinite number of substitution groups, each of which is distinct from the infinite number of groups that are obtained by the same process from any substitution that is not similar to s_1.

Mr. Kuhn's paper relates to imprimitive groups that contain more than one set of systems of imprimitivity. Let G denote any regular group. It is proved that the number of sets of systems of imprimitivity that can be found for G is equal to the number of its subgroups (not including identity). The cyclical group of order p^2 is therefore the only regular group that contains just one set of systems of imprimitivity. Any set of systems that corresponds to a subgroup that contains no invariant subgroup of G besides identity is permuted by the substitutions of G according to a transitive group that is simply isomorphic to G; any other set is permuted according to a transitive group that has a $(1, \alpha)$ isomorphism to G, α being greater than unity.

Let G' denote any imprimitive group that has more than one set of systems of imprimitivity. When the substitutions of G' permute all its sets of systems according to primitive groups and when none of the heads is identity, then

(1) No two of the heads of G' can contain a common substitution besides identity, and hence

(2) Each substitution of any other given head is commutative to each substitution of any other head.

(3) Each head contains at least one substitution whose degree equals the degree of G', and

(4) Each head is formed by establishing a simple isomorphism among its transitive constituents.

After a brief statement of the principles of the stereographic projection Professor Penfield exhibited a number of devices for plotting. One of these consisted of sheets upon which were printed a graduated circle of 14 cm. diameter and four scales, as follows: No. 1, giving the radii of stereographically projected arcs of great circles. No. 2, giving the radii of stereographically projected arcs of vertical small circles. No. 3, giving the degrees of a vertical circle stereographically projected upon a diameter. No. 4, giving the radius of the circle divided into one hundred parts. Other devices consisted of two protractors, one having along its base line a scale giving the degrees of a circle stereographically projected, the other used in measuring and for that purpose printed on transparent celluloid, giving the arcs of vertical small circles stereographically projected. There were also exhibited devices of a similar nature, but made up on a large scale for blackboard demonstrations.

Right and oblique angled triangles were solved graphically on the blackboard, and the graphical solution of similar problems carefully executed on the printed sheets were exhibited, indicating that within a circle of 14 cm. diameter computations could be made with an average error not exceeding seven minutes.

There were also exhibited a patented arm protractor for drawing and measuring plane angles, and goniometers especially devised for measuring crystals.

Special emphasis was laid upon the importance of graphical methods for making the studies of plane and spherical trigonometry and geometry more real to students, as a means of checking numerical calculations, and as a rapid method for obtaining approximate results.

The subject of Miss Schottenfels's paper was suggested by the investigation of Professor E. H. Moore for the function $\frac{\sin \pi z}{\pi}$ in the *Annals of Mathematics*, volume 9, January, 1895. A set of definitional functional properties are obtained for the function $f(z) = \frac{\tan \pi z}{\pi}$, and it is shown that there is one and only one function $f(z)$ possessing these properties. The external exponential factor is then determined in the expression of the function as a Weierstrassian infinite product of primary factors.

Professor Smith's paper takes up the relation of the geometry of reciprocal radii to the geometry within a spherical complex, as determined by the single isomorphism of the corresponding groups, viz., these groups are transformed into each other by contact transformation, whose properties were defined and studied by the author in the paper published in the *Transactions*, October, 1900, "On surfaces enveloped by spheres belonging to a linear spherical complex."

In a paper, soon to be published in the *American Journal of Mathematics*, Dr. Wilczynski has shown that the most general point transformation which converts a general system of n linear homogeneous differential equations into another of the same form and order is

$$(1) \qquad x = f(\xi), \quad y_k = \sum_{l=1}^{n} a_{kl}(\xi)\eta_l, \qquad (k = 1, 2, \cdots, n),$$

where $f(\xi)$ and $a_{kl}(\xi)$ are arbitrary functions of ξ.

Functions of the coefficients and of their derivatives which have the same value for the original as for the transformed system are called invariants. Covariants are such invariant functions involving also $y_1, \cdots, y_n, y_1', \cdots, y_n'$, etc.

Equations (1) define an infinite group G in the $n + 1$ variables $x, y_1, \cdots, y_n$. A subgroup of this is that for which x remains unchanged and only $y_1, \cdots, y_n$ are transformed. Functions of the coefficients invariant under this subgroup are called *seminvariants*. These can be found by setting up, by the method of infinitesimal transformations, the partial differential equations which they satisfy. The invariants are functions of the seminvariants.

The actual computation of the seminvariants and invariants is carried out only for a system of two equations each of the second order, *i. e.*, for the system

$$(2) \qquad y_i'' + p_{i1}y_1' + p_{i2}y_2' + q_{i1}y_1 + q_{i2}y_2 = 0, \qquad (i=1,2),$$

but the methods employed admit of generalization.

If we put

$$(3) \qquad \begin{aligned} u_{11} &= 2p_{11}' - 4q_{11} + p_{11}^2 + p_{12}p_{21}, \\ u_{12} &= 2p_{12}' - 4q_{12} + p_{12}(p_{11} + p_{22}), \\ u_{21} &= 2p_{21}' - 4q_{21} + p_{21}(p_{11} + p_{22}), \\ u_{22} &= 2p_{22}' - 4q_{22} + p_{22}^2 + p_{12}p_{21}, \end{aligned}$$

then two seminvariants are

$$(4) \qquad I = u_{11} + u_{22}, \quad J = u_{11}u_{22} - u_{12}u_{21}.$$

Four quantities v_{ik} are now formed, cogredient with the quantities u_{ik}, viz.,

$$(5) \qquad \begin{aligned} v_{11} &= 2u_{11}' + p_{12}u_{21} - p_{21}u_{12}, \\ v_{12} &= 2u_{12}' + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}), \\ v_{21} &= 2u_{21}' - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}), \\ v_{22} &= 2u_{22}' - p_{12}u_{21} + p_{21}u_{12}, \end{aligned}$$

and a new independent seminvariant is

$$(6) \qquad K = v_{11}v_{22} - v_{12}v_{21}.$$

A third set of quantities w_{ik} is formed from v_{ik} and p_{ik}, in exactly the same way as v_{ik} from u_{ik} and p_{ik} in (5), i. e.,

$$(7) \qquad \begin{gathered} w_{11} = 2v_{11}' + p_{12}v_{21} - p_{21}v_{12}, \\ \text{etc., etc.;} \end{gathered}$$

and since the v_{ik}'s were cogredient with the u_{ik}'s, so are the w_{ik}'s. Then

$$(8) \qquad L = w_{11}w_{22} - w_{12}w_{21}$$

is a new seminvariant.

All seminvariants are functions of I, J, K, L, and their derivatives.

Some of the invariants are

$$\theta_4 = I^2 - 4J \quad ,$$

$$\theta_6 = 2I(I^2 - 4J) + 5\,(K - I'^2) + 4(K - 2J'' + II'').$$

etc., these being of weight 4 and 6 respectively.

If the system of equations (2) is equivalent to a single linear differential equation of the second order, all of the invariants vanish, and conversely.

Any system of form (2) can be transformed into another for which $p_{12} = 0$. This is called the semicanonical form of the system. The subgroup G' of G which leaves this form unchanged is examined. But we can fulfill the further condition $q_{11} + q_{22} = 0$, and a system for which both $p_{12} = 0$ and $q_{11} + q_{22} = 0$, is said to be in the canonical form. The subgroup G'' of G' which leaves this unchanged is a finite group of very simple form, and has some additional invariants of the form called quadri-derivatives by Forsyth.

Only a few simple results about covariants are mentioned. This and further generalizations are left for a future paper.

F. N. COLE.

COLUMBIA UNIVERSITY.

ON LINEAR DEPENDENCE OF FUNCTIONS OF ONE VARIABLE.

BY PROFESSOR MAXIME BÔCHER

It is ordinarily stated that the identical vanishing of the determinant

$$D = \begin{vmatrix} u_1 & u_2 & \cdot\ u_n \\ u_1' & u_2' & \cdots u_n' \\ \multicolumn{3}{c}{\dotfill} \\ \multicolumn{3}{c}{\dotfill} \\ u_1^{[n-1]} & u_2^{[n-1]} & \cdots u_n^{[n-1]} \end{vmatrix}$$

is a sufficient* condition for the linear dependence of the functions $u_1(x), u_2(x), \cdots, u_n(x)$. This is perfectly true if the u's are analytic functions of the complex variable x. *This condition is however no longer sufficient if we are dealing with functions of a real variable,* even though these functions possess derivatives of all orders for every real value of x. The truth of this statement will be seen from the example of two functions u_1 and u_2 defined as follows:

$$u_1 = \begin{cases} e^{-1/x^2} & (x + 0) \\ 0 & (x = 0) \end{cases} \qquad u_2 = \begin{cases} e^{-1/x^2} & (x > 0) \\ 0 & (x = 0) \\ 2e^{-1/x^2} & (x < 0) \end{cases}$$

---

* It is of course a necessary condition provided the u's have derivatives of the first $n - 1$ orders.

Similar examples for a larger number of functions can readily be built.

The following theorem however is true:

If $u_1, u_2, \cdots, u_n$ *are single valued functions of the real variable* x *defined at every point of a certain interval and having at every point of this interval derivatives of the first* $n-1$ *orders, and if it is possible to strike out the last row and one of the columns of the determinant* D *in such a way that there is no point of the interval in question at which the remaining determinant and its derivative both vanish, then if* D *vanishes at every point of the interval, the functions* $u_1, u_2, \cdots, u_n$ *will be linearly dependent throughout this interval.*

This theorem can be readily proved by a slight extension of the method given for instance by Heffter in his book on linear differential equations p. 233.

EMS, GERMANY,
September 15, 1900.

[*Note added November 2, 1900:* I have just found in Pascal's book on determinants a reference to three papers by Peano (*Mathesis*, vol. 9 (1889), p. 75 and p. 110; *Rend. d. Accad. d. Lincei*, ser. 5, vol. 6 (1897), 1° sem., p. 413), in which the question which I have here considered is taken up. My result is however different from Peano's, which states that the identical vanishing of D is a sufficient condition for linear dependence, provided there is no point at which the first minors corresponding to the elements of the last column all vanish.]

REPORT ON THE GROUPS OF AN INFINITE ORDER.

BY DR. G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, New York, June 28, 1900.)

VARIOUS terms have been employed to designate the smallest elements of which any abstract group is composed. Cayley has called them symbols,* or symbols of operation. Dyck and many others have called them operations† or operators. Frobenius and others have called them elements.‡ In what follows we shall employ the last one of

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*Cayley, *Phil. Magazine*, vol. 7 (1854), p. 41.

†Dyck, *Math. Annalen*, vol. 20 (1882), p. 1.

‡Frobenius, *Crelle*, vol. 86 (1879), p. 218.

these terms. According to the early definitions of an abstract group a set of distinct elements becomes a group with respect to a given law of combination when it has the property that no new element is obtained by combining the elements of the set in every possible manner according to this law of combination.* In recent years the following additional conditions have been imposed:

(1) The elements must be associative when combined according to the given law. (2) From either of the two equations $s_1s_2 = s_1's_2$, $s_2s_1 = s_2s_1'$ it follows that $s_1 = s_1'$. (3) If any two of the elements of the equation $s_1s_2 = s_3$ are in the given set there is one and only one element in the set which may be used for the third. This definition is much more explicit than Cayley's dictum† "a group is defined by the law of combination of its symbols," since it is easy to give definite laws of combination of a set of distinct elements which do not form a group. Every particular group is, however, defined by the special laws of combination of its symbols—by its own multiplication table.

One of the best known examples of a group of an indefinitely large order is furnished by the totality of finite integers when they are combined by addition. If we denote by s^{-a} the inverse of s^a, it is clear that the negative exponents of elements may be used to indicate subtraction and that the identical element is $s^0 = 0$, when the elements of a group are combined by addition. In general, if s represents any integer, the exponent of s will indicate the number of times this number is to be added or subtracted. The given group of indefinitely large order is evidently generated by $s_1 = 1$ as well as by $s_2 = s_1^{-1} = -1$, but by no other one of its elements. While $\varphi(m)$ of the elements of any cyclical group of a given finite order m are generators of the group we have here a cyclical group of an indefinitely large order which contains only two generating elements. All the multiples of any integer whose absolute value exceeds unity constitute a subgroup of this group and each one of these subgroups is of an indefinitely large order and involves only two elements that generate it.

In order to obtain a subgroup of a finite order m we may take the complete series of smallest positive residues mod. m $(0, 1, 2, 3, \cdots, m-1)$. When we add any two numbers of this series and use the smallest positive residue in place of the sum we evidently obtain some number of this series.

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*Cayley, loc. cit.; Lie, Le centenaire de l'École normale, 1895, p. 486; Klein, Vorlesungen über das Ikosaeder, 1884, p. 5.

†Cayley, *Amer. Jour. of Math.*, vol. 1 (1878), p. 51.

As the other group conditions are satisfied we have here a cyclical group of order m which is generated by any one of the $\varphi(m)$ numbers that are not greater than m and prime to it. Hence we observe that the finite integers, when combined by addition, furnish very elementary examples of groups of any given finite orders as well as of groups of indefinitely large orders.

Since x cannot be an integer in such an equation as $\delta x = 19$, it follows that the integers do not constitute a group with respect to multiplication.* The finite commensurable numbers which exceed zero, on the contrary, constitute such a group. Those which are less than zero do not constitute a group; but, together with those that exceed zero, they clearly constitute a group with respect to the given operation. This group of infinite order is included in the one formed by all the real numbers with the exception of 0 and ∞. The latter group is, in turn, included in the abelian group formed by multiplying together all the complex numbers whose absolute value is neither 0 nor ∞. The finite commensurable numbers, including 0, evidently constitute a group of infinite order when they are combined by addition and this group is included in the one formed by all the real finite numbers. The totality of complex numbers of finite absolute value constitute a group, with respect to addition, of which the preceding is a subgroup.

Although all the integers do not form a group with respect to multiplication, yet it is clear that all the powers of any finite integer, with the exception of 0 and 1, constitute a group of an indefinitely large order with respect to this operation. In order to obtain a group of a finite order by multiplying integers together we may take the $\varphi(m)$ positive numbers which are less than m and prime to it and replace their products by the least positive residues mod. m. We thus obtain a very important class of abelian groups which has been studied recently by Weber and others.†

While the fundamental operations addition, multiplication, and their inverses furnish such lucid examples of groups of an infinite as well as of a finite order, yet it is not customary to claim that the theory of groups is as old as these elementary operations. At least one group property of these operators must have been observed very early; viz., that the successive application of two elements is equivalent to the single application of some one of them. While this concept is of great theoretic interest yet it did not enter explicitly into the early use of these fundamental operations and hence it may perhaps be said to have played a secondary rôle in the developments along this line.

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* Weber, Lehrbuch der Algebra, 1899, p. 4; cf. ibid., 1896, p. 54.

† Weber, Lehrbuch der Algebra, 1899, p. 60. Cf. Burnside, Theory of groups of a finite order, 1897, p 239.

The concept of group of an infinite order appears in many other early developments but the earliest extensive paper in which this concept occupies a very prominent place seems to be the "Mémoire sur les groupes de mouvements," by Jordan, which was published in the *Annali di Matematica*, vol. 2 (1868).* No general definition of a group occurs in this memoir but on page 181 a "définition caractéristique d'un groupe de mouvements" is given, which is in accord with the present definition. An abstract of this memoir was published somewhat earlier in the Paris *Comptes rendus*, vol. 65 (1867). In the beginning of the memoir Jordan points out that the determination of all the groups of movements is equivalent to the determination of all the possible systems of molecules which can be superposed in different positions.

Bravais had studied particular cases of this question at a much earlier date from the latter point of view,† and applied his results in the study of the structure of crystals. Since Jordan's memoir was inspired by the work of Bravais it might perhaps be said that the theory of groups of an infinite order has its source in the theory of the structure of crystals. However, the group concept does not occupy as prominent a place in the works of Bravais as in the mentioned memoir of Jordan. We proceed to give an elementary exposition of the first part of this memoir with a view to giving an accurate idea of its contents. After some preliminary remarks and a brief study of the composition of movements the author proceeds to determine all the groups of translations in the following manner:

The lengths and the directions of all the translations of the group are represented by line segments starting from a general point P. It is first assumed that all of these segments are of finite lengths and the length of the shortest segment L_1 (or one of the shortest segments, if there is more than one of minimum length) is represented by l. The line through L_1 is taken for the x-axis. An indefinite number of segments of length l are laid off on the x-axis starting from P. The point P can evidently be transformed into any one of these points of division on the x-axis by means of some translation of the group. If P could be translated into any other point on the x-axis, its distance from P would be $ml \pm r$, m being an integer and $0 < r \leqq l/2$. If we combine this translation with the translation through $-ml$, which is also in the group, we obtain a translation whose length cannot exceed $l/2$. As this is contrary to the hypothesis, it follows that the points of division can be transformed into none of the points on the x-axis except points of division, by all the translations of the group.

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* Cf. Fricke und Klein, Automorphe Functionen, vol. I, 1897, p. 12. Klein seems to have been the first to make prominent use of the concept of discontinuous groups of an infinite order in function theory; cf. Dyck, *Math Annalen*, vol. 20 (1882), p. 2, footnote.

† Bravais, *Liouville's Journal*, vol. 14 (1849), p. 167.

After proving that all the translations parallel to the x-axis are generated by a single translation the author proceeds to determine all the possible translations in the plane and proves that all of these are generated by two translations which make a finite angle with each other. This is done as follows: Among the given line segments starting from P we take one L_2 whose extremity is at a minimum distance from the x-axis and represent this distance by δ. The first object is to prove that δ must be finite. We may assume that the projection a of L_2 upon the x-axis does not exceed $\frac{1}{2} l$ since the group involves translations of length l parallel to the x-axis. Since the length of L_2 cannot be less than l, we have

$$l^2 \leqq \sqrt{a^2 + \delta^2} \leqq \sqrt{\frac{l^2}{4} + \delta^2}, \text{ or } \delta^2 \geqq \frac{3}{4} l^2.$$

On the line through L_2 an indefinite number of segments of the same length as L_2 are laid off, starting from P. Through the points of division lines are drawn parallel to the x-axis and through the given points of division of the x-axis lines parallel to L_2 are drawn. The plane is thus divided into parallelograms, as is usually done in the study of doubly periodic functions. It is then proved that the vertices of these parallelograms can be transformed only among themselves by all the translations of the group and hence there is no finite translation in the plane determined by L_1 and L_2 except those which are generated by the translations represented by the segments L_1 and L_2. From this it follows that *there are two and only two groups of translations, involving only finite translations, in a given plane.* In one of these groups all the translations are parallel to a given line; in the other they are generated by two translations making a finite angle with each other.

If a group contains any finite translations in addition to those given above they cannot lie in the plane determined by L_1 and L_2. We choose one of them, having a minimum projection β upon the normal to the given plane, and represent it by L_3. The length of β may be proved to be finite by the following method. The projection of L_3 upon the given plane may be assumed to lie in any one of the four parallelograms which have P for a common vertex. This parallelogram is so chosen that the distance from the x-axis to the terminus of the given projection is $\leqq \delta/2$. Since L_3 cannot terminate closer to the x-axis than L_2 it follows that

$$\delta \leqq \sqrt{\beta^2 + \frac{\delta^2}{4}}, \quad \text{or} \quad \beta^2 \geqq \frac{3}{4}\delta^2.$$

Space is now divided into parallelepipeds whose vertices are the points into which P may be translated by all the operators of the group generated by the translations represented by L_1, L_2, L_3. It is readily proved that a vertex of such a parallelepiped cannot be transformed into any point except a vertex by means of the translations of the group. This is therefore the largest group composed entirely of finite translations in ordinary space and the two groups mentioned above are subgroups of this one. Every point of space which is not a vertex of the given system of parallelepipeds will clearly be transformed into a vertex of a similar system of parallelepipeds by the translations of this group.

There are six groups of translations that involve indefinitely small translations. The first of these is generated by a single infinitesimal translation, the second by an infinitesimal and a finite translation not in the same dirección, the third by an infinitesimal and two finite translations the three translations being not co-planar, the fourth is generated by two infinitesimal translations not in the same direction, the fifth by two infinitesimal and one finite translation not in the same plane, and the sixth by three infinitesimal translations not in the same plane. The last of these groups includes all the others as subgroups and each one of these nine groups is of an infinite order.

After the groups which involve only translations have been determined the author proceeds to the groups which are composed of rotations only. Here the matter becomes somewhat more difficult. The first step is to prove that all the axes of rotation must go through the same point. This is done as follows: Let R_1, R_2 be two such rotations. Their combination $R_1 R_2 = R_3$ is a new rotation around some axis A_3. Let x be a point on A_3. R_1 will bring x to some new point x_1. Hence the axis of R_1 must lie on the plane perpendicular to the segment joining x and x_1 at its middle point. Since R_2 must transform x_1 into x its axis must lie in the same plane. Hence the axes of any two rotations must go through the same point. Since this point is transformed into itself by the two rotations R_1, R_2 it must also lie on A_3. As A_3 passes through the point x, which is not in the plane of the axes of R_1 and R_2, the three axes of rotation cannot lie in the same plane. Since any other axis of rotation has to meet each of these three lines it must go through their common point, as the lines are not in the same plane. This proves that all the axes of a group composed entirely of rotations must be concurrent.

It is not difficult to see that there are at least eight groups which are composed of rotations only. The first is generated by the rotation through an angle $2\pi/n$, n being any integer. The second is generated by rotating through an infinitesimal angle. The third and fourth are generated by adding to each of the preceding a rotation through 180° around an axis perpendicular to the given one. The fifth, sixth and seventh are the well known rotations which transform the regular tetrahedron, the cube, and the regular icosahedron[*] into themselves. These groups are respectively of order 12, 24, and 60, and may be represented as the alternating group of four letters, the symmetric group of four letters, and the alternating group of five letters respectively. The eighth is composed of all the possible rotations around a point. The proof that there is no other group of rotations is somewhat lengthy, and we shall not enter upon it.

The greater part of the memoir under consideration is devoted to groups which involve both rotations and translations. These have recently been studied by Schönflies, who published his results in two well-known memoirs.[†] It may be remarked that Jordan's memoir is not entirely free from errors. Sohncke seems to have been the first to observe that an important class of groups was overlooked in the memoirs mentioned.[‡] The determination of all the continuous groups of euclidean and non-euclidean movements in a space of three dimensions was first published by Lie.

Although groups of an infinite order were studied before Lie and Klein began to publish on this subject, yet they were the first to bring them prominently before the public.* Their investigations along this line are so far reaching and so numerous as to deserve a report by themselves. Happily the German Mathematical Association has arranged for a report on Lie's work, which is expected to be presented at the meeting of next year (1901). The report is to be prepared by Engel assisted by Scheffers, Schur, and Kowalewski —men who are in the best possible position to prepare an extensive and reliable report in reference to the work of this great Norwegian mathematician. In view of this fact we shall not consider the works of Lie and his followers in the present report.

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[*] Cf Klein, Vorlesungen uber das Ikosaeder, pp. 1–19.

[†] *Math. Annalen*, vol., 28, pp. 319–342 ; vol. 29, pp. 50–80 ; cf. Schönflies, Chicago Mathematical Congress Papers, 1896, p. 341.

[‡] Theorie der Krystallstruktur, 1879, p. 26.

In 1882 two remarkable memoirs, dealing with groups of an infinite order, appeared. One of these was published by Dyck in the *Mathematische Annalen*, volume 20, and the other by Poincaré in the *Acta Mathematica*, volume 1. These articles have received so much attention that it seems unnecessary to give an outline of them. Burnside has devoted Chapter XII. of his Theory of groups of a finite order (1897) to an exposition of Dyck's memoirs, while many parts of Poincaré's memoir are elucidated in the work on Automorphic Functions by Fricke and Klein, which also appeared in 1897.

This last work makes a clear distinction between continuous groups and groups which contain infinitesimal transformations. It is easy to prove that a continuous group always involves infinitesimal transformations, for if T is any transformation of the group and T' the transformation obtained by using parameters which differ by an infinitesimal from those of T then $T' T^{-1}$ is infinitesimal. However the existence of infinitesimal transformations does not prove the group continuous. Attention is called† to the fact that in Lie-Engel's work on Transformationsgruppen and in Poincaré's memoir not sufficient emphasis is laid upon this distinction. The work of Fricke and Klein on the group-theoretic foundation of automorphic functions is doubtless the best work on discontinuous groups of an infinite order that has yet been published. The direct object of this work is the investigation of the groups of linear substitutions in one variable which do not contain any infinitesimal substitution.

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* Cf. Klein, Erlangen Programme (1872), BULLETIN, vol. 2 (1893), p. 215. Lie's contributions are systematically treated in the three volumes of his Theorie der Transformationsgruppen (1888–93); Differentialgleichungen (1891); Continuierliche Gruppen (1893); Berührungstransformationen, vol. 1 (1896); Klein's Einleitung in die hohere Geometrie (1893); etc. A considerable part of Klein's work along this line is contained in the two volumes of his Modulfunctionen (1890–92), and in the Automorphe Functionen, vol. 1 (1897). Numerous publications by Picard and Poincaré have also contributed very much towards the general interest in this subject. Picard was the first to develop the theory of linear differential equations parallel with the Galois theory of algebraic equations. Vessiot, Drach, and others have perfected this theory in certain directions.

† Page 65.

While in Lie's theory the parameters of the product of two transformations are functions of the parameters only of these transformations, Halphen has considered groups in which these parameters are functions both of the variables and of the parameters of the two transformations that are multiplied together. Lie pronounced this group concept trivial and said that it had not found any application.* He expressed the opinion that the definition of group of transformations had perhaps reached a definite form, but that it was not certain that the general notion of group had as yet assumed such a form.†

Study and Engel seem to have discovered independently ‡ that the special linear homogeneous group of the plane contains transformations which are not generated by any infinitesimal transformations of the group, as was apparently assumed by Lie.§ Such transformations have been called *singular transformations* and they have received considerable attention in recent years. The two memoirs by Engel have been followed by numerous other memoirs especially by Taber,‖ who established the existence of singular transformations in the group of orthogonal substitutions in n variables, $n > 3$, and also in the group of linear automophic transformations of a general bilinear form. He has also determined many properties of groups that contain such transformations.

One of his students, Mr. Rettger, has recently investigated ¶ the two and three parameter subgroups of the general projective group in two variables and of the general linear homogeneous group in three variables with regard to singular transformations, and proved that singular transformations occur among the transformations of many of these groups. A list of such subgroups is given at the end of the article. Many of the results of this article are contained in an earlier note by the same author in the *Proceedings of the American Academy*, vol. 33 (1898), p. 493.

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* Theorie der Transformationsgruppen, vol. 3 (1893), p. 19; cf. *Amer. Jour. of Math.*, vol. 11 (1889), p. 182.

† *Le centenaire de l'Ecole normale*, 1895, p. 486.

‡ Engel, *Leipziger Berichte*, vol. 44 (1892), pp. 277–296 and vol. 45 (1893), pp. 659–696; cf. BULLETIN, vol. 3 (1893), p. 66.

§ Cf. Taber, BULLETIN, vol. 6 (1900), p. 199.

‖ Taber, *Amer. Jour. of Math*, vol. 16, p. 130; *Proc. of the Lond. Mat. Soc.*, vol. 26 (1895), p. 364; *Math. Annalen*, vol. 46, p 561; BULLETIN, vol. 6 (1900), p. 199. Numerous other references are contained in these articles.

¶ Rettger, *Amer. Jour. of Math.*, vol. 22 (1900), p. 60.

Besides the groups which are explicitly of an infinite order there is a large class of groups whose orders depend upon one or more parameters, which are generally assumed to be finite but may also be regarded as infinitely large. To this class belong the symmetric and the alternating groups of degree n,* the cyclical groups of order n, the metacyclic and the semi-metacylic groups, of orders $p(p-1)$ and $\frac{1}{2}p(p-1)$ respectively, etc. Such groups are, however, generally classed with the groups of a finite order. We observed above that Burnside devoted a small part of his theory of groups of finite order to the study of groups of an infinite order. In the second volume of Weber's Algebra these groups receive much more attention. Considerable portions of the chapters on linear groups are devoted to considerations which explicitly relate to groups of an infinite order. The object seems to be to devote some space to discussions in which the order is not restricted and then to proceed to the special cases where the order is finite.

The theory of groups of an infinite order is closely related to that of a finite order and the development of the former has been greatly influenced by the latter. The latter, in turn, has contributed very much towards the interest and importance of the former, and, in several instances, it has led to important new developments in this theory; *e. g.*, Maillet has shown † how the concept of parameter groups, as used by Lie, can be employed in the theory of substitution groups and has made use of this concept to obtain several general theorems on simply isomorphic groups. In a recent article,‡ Loewy has established the fundamental theorem that a linear substitution group of infinite order which contains at least one substitution whose characteristic equation has no equal roots, must always contain a substitution whose order exceeds any given number. The presence of such a substitution is therefore a necessary and sufficient condition that the order of a linear group is infinite.

CORNELL UNIVERSITY,
June, 1900.

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* Dyck, *Math. Annalen*, vol. 22 (1883), p. 72.

† Maillet, *Ann. di Matematica*, vol 23 (1895), p. 199.

‡ Loewy, *Math. Annalen*, vol. 53 (1900), p. 225.

THE STRENGTH OF MATERIALS.

The Strength of Materials. By J. A. Ewing, M.A., B.Sc., F.R.S., M.Inst.C.E. Professor of Mechanism and Applied Mechanics in the University of Cambridge, Fellow of King's College, Cambridge. Cambridge, The University Press, 1899. Pp. vii–xii + 1–246, with 150 illustrations in the text.

This work is intended to supply to students "in modern schools of Engineering" a "knowledge of the Strength of Materials and of its application in design * * * which to be effective must be supplemented by laboratory and drawing-office work." It is desirable that the mathematical elastician should learn what light modern experimental research throws on the validity of the hypotheses at the basis of the mathematical theory, and also that he should know what parts of his subject possess most interest for his more practical contemporaries. On the other hand, it is conceivable that the practical man may derive some advantage from realizing how the mathematical treatment which passes current in everyday life strikes the mathematician. This review thus naturally divides itself into two principal parts, the first dealing with the more experimental portions of the book, the second with the mathematical methods. A preliminary description of the contents of the book will facilitate comprehension.

After a brief preface and a table of contents, pp. vii–xii, chaps. I and II, pp. 1–23, define stress and strain, explain their simpler common types, and treat of the ordinary moduli, or elastic constants, for *isotropic* material. Chap. III, pp. 24–58, treats of ultimate strength and non-elastic strain, dealing mainly with simple tension and compression, describes the phenomena presented during the loading of steel and iron up to rupture, and discusses experiments on the effects of rest or of heating after over-straining. It also gives an account of Wohler's experiments on the "weakening" of material under very frequent repetitions of loading and unloading. Chap. IV, pp. 59–95, describes—with excellent illustrations—a number of testing machines and instruments for measuring extensions and compressions, discusses some methods of determining the modulus of rigidity and gives some numerical results.

Chaps. V, VI, and VII, pp. 96–153, deal mainly with the application of the ordinary Bernoulli-Euler mathematical theory to beams. Analytical methods are generally followed, but the ordinary graphic methods are also illustrated. Chap. VIII, pp. 154–170, treating of "frames," may be regarded as part of a text book on ordinary statics. Chap. IX, pp. 171–186, treats of struts and columns, first on Euler's mathematical theory, and then from the more empirical standpoint of Gordon or Rankine's formula. Chap. X, pp. 187–203, is denominated "Torsion of Shafts." It is however not confined to this, but touches upon the theory of spiral springs and the "whirling" of rotating shafts. Chap. XI, pp. 204–217, treats of shells and thick cylinders under pressure, with a discussion of rotating rings and thin discs. Chap. XII, pp. 218–237, is devoted to "Hanging Chains and Arched Ribs," including parabolic chains, the common catenary, etc. An appendix, pp. 239–242, gives numerical data as to strengths and densities of ordinary building materials ; and an index occupies pp. 243–246.

Experiment.—The fundamental fact underlying the ordinary mathematical theory, and therefore all ordinary girder theory, whether its outward form be graphical or analytical, is Hooke's law that stress is proportional to strain, or—as Professor Pearson puts it—that the stress-strain relation is *linear*. On this point we are told (Art. 15) "A material is elastic * * * if the strain disappears when the stress is removed. Strain which persists after the stress * * * is removed is called permanent set. * * * Actual materials are in general nearly perfectly elastic with regard to small stresses * * * if the applied stress is less than a certain limit, the strain * * * disappears wholly or almost wholly when the stress is removed. * * * The limits of stress within which strain is wholly or almost wholly elastic are called limits of elasticity * * *. (Art. 16) Within the limits of elasticity the strain * * * is proportional to the stress." Later we have the following statement (Art. 32): "Within these limits (of elasticity) we may without serious inaccuracy take the strain as * * * proportional to the stress and as disappearing when the stress is removed. Strictly speaking, absolute proportionality of strain to stress is never found, and probably there is no stress, however small, that does not produce some permanent effect. There is always some slight *hysteresis* or lagging in the relation of strain to stress. * * * But in general this imperfection of elasticity is so slight that * * * up to a certain limit, which is in general pretty well defined, Hooke's Law may be taken as substantially accurate."

A student when reading Art. 15 would be apt to conclude that the only two things known to exist are *elastic strain* disappearing simultaneously with the removal of the stress, and *set* persisting permanently. In Act. 32 he seems, however, expected to distinguish between set and hysteresis (a term which seems equivalent to "Elastische Nachwirkung" or what Pearson calls "Elastic after-strain"). It is to be regretted that the author does not bring out more clearly the theoretical distinction between set which persists and strain which gradually disappears, especially as the practical discrimination between them is so difficult.

Another question is whether the author is well advised in assuming—as he seems tacitly to do—that failure of Hooke's law necessarily implies the existence of hysteresis, or set, and conversely. It is at least conceivable that materials exist in which the relation between stress and *elastic* strain is not linear. Experiments by Hodgkinson, Kupffer, and others have sometimes been supposed to show that cast iron is such a material. Our author refers to Hodgkinson's experiments (Art. 38), but not in such a way as to show whether he is aware of the interpretation put upon them by others (see Todhunter and Pearson's "History of Elasticity" Vol. 1, Note D, p. 891; Vol. 2, Art. 793, etc.). The subject seems one which might repay experimental re-investigation with modern appliances.

The next question that calls for consideration is that which gives its name to the book. In Art. 33 we are told "The load which suffices to cause rupture measures the ultimate strength of the piece." At this stage, however, a difficulty comes in. Engineering calculations are almost all ultimately based on Hooke's law. But when the stress is carried beyond the elastic limit the departure from the law begins to be serious, and with the approach of rupture the law is often a travesty of the facts. If then calculation supplies values for the stresses or strains, at any point of a structure, such as answer to the breaking load, the only legitimate conclusion is that the physical basis for the calculation is not satisfactory. It would certainly be prudent in such a case to conclude that the load contemplated is excessive, or badly distributed; but how excessive it may be, or what would happen if it were applied, lies outside the range of the ordinary theory. From the theoretical standpoint the most logical course would seem to be to consider not the breaking load but that answering to the elastic limit. According to our author, the elastic limit is usually pretty clearly defined in most building materials; and, supposing this to be true, there would not appear to be serious objection on the practical side to focussing attention on it rather than on the breaking load. It appears, however, to be the invariable or almost invariable custom amongst practical men to make use ostensibly of the breaking load modified by so-called "factors of safety." After giving the definition

factor of safety = (ultimate strength) ÷ (extreme working stress),

our author adds (Art. 34): "The rational use of a factor of safety * * * results * * * in preventing waste of material locally by making the margin of strength equal for all parts." Perhaps a more logical way of presenting the facts would be to say that what the engineer puts before him is the "extreme working stress" which he gets by dividing his ultimate strength, *i. e.*, breaking load, by a certain number which he calls a factor of safety. Supposing the working stress thus arrived at not to strain the material beyond the elastic limit, there is no objection on mathematical grounds to calculations in which it plays a part.

There still, however, remains a difficulty; for how are we to determine the factor of safety? On this point we are told in Art. 34: "The choice of a factor * * * depends on many considerations, such as the probable accuracy of the estimated loads and also that of the theory * * * the possible effects of bad workmanship, * * * the variability or uniformity of the load. * * * The factor * * * also serves to provide for * * * incidental shocks. * * * The factor * * * is rarely less than 3, it is very commonly 4 or 5, and it is sometimes as much (in machines) as 10 or 12." The factor of safety covers, in short, uncertainty in the material, unforeseen contingencies from loading or weather, and gaps in the engineer's physical or mathematical knowledge. So long as all these sources of uncertainty are lumped together, "a rational use of a factor of safety" seems rather much to hope for. One of the things most likely to narrow the range of uncertainties is a knowledge of what really determines the changes that take place in the material as the load is raised. There are various theories bearing on this point. The three principal believe that in isotropic materials the crux is: (1) the maximum principal stress, (2) the maximum stress-difference (or difference between the greatest and least principal stresses), (3) the greatest principal strain. Of these theories the first—employed by Lamé and others—is now hardly supported by any theoretical elastician; the second—suggested by Tresca's experiments on the flow of metals—has been used by Prof. G. H. Darwin and others; the third was that advocated by St. Venant. Our author refers explicitly to no theory. In dealing with ultimate strengths he invariably speaks of the *stress*, and his language in many places (e. g., Art. 8 where he says "the greatest principal stress measures the greatest intensity which the material has to bear") certainly suggests that he supports the first or greatest stress theory of rupture. In general, however, he really avoids committing himself. When a piece of given material, of given size and shape, is loaded in a particular way, so long as Hooke's law holds, doubling the load doubles alike every stress and strain; thus the condition of the material can be defined by specifying the magnitude of any one convenient stress or strain. In such a case the laying down a limiting *stress* does not commit one to any theory of rupture. If, for instance, we take a uniform bar under simple tension S, the greatest stress and the maximum stress-difference are both S, and the greatest strain is S/E, where E is a constant for the material (Young's modulus). Thus we may specify that the working stress shall not exceed $\bar{S}$, and yet believe that the critical thing is that the maximum stress-difference shall not exceed $\bar{S}$, or the greatest strain shall not exceed $\bar{S}/E$. Our author refers to separate limiting stresses for simple tension, simple compression, and shearing; and if the simple states were those that always existed, one might get along fairly comfortably without any theory of rupture. As matters actually stand, however, empiricism is under the disadvantage that absolutely simple tension, compression, or shearing are rather ideal states, and are only approximated to even in the testing machine. In some of the problems treated in this book the stress system is not simple, and, as we shall see later, the conclusions drawn as to the strength are not always reliable unless the maximum stress theory be really true.

Nature of material.—In Arts. 17 to 19, Young's and the rigidity moduli are defined in a way applicable only to isotropic materials, while isotropy is first mentioned, and then only incidentally, in Art. 20. In Art. 26, where the stress-strain relations are formulated, it is indeed mentioned that isotropy is assumed, but nothing is said as to the general form of the relations in non-isotropic materials. In Art. 33 there are fuller references to differences between isotropic and non-isotropic materials, but only from the point of view of ultimate strength. The same article tells us that wrought iron is not isotropic, and data confirming this statement appear in Table III, p. 241. Yet in Table IV, p. 241, there appear values of Young's and the rigidity moduli for wrought iron, as if it were isotropic, and we even have a Young's modulus given for "timber" without any specification of direction. All the mathematical calculations in the book, which depend in any way on the precise form of the stress-strain relations, tacitly assume isotropy; and no attempt is made to ascertain what modification is necessary to render the results applicable to non-isotropic material. This assumption of isotropy is by no means peculiar to the present work. Rankine and St. Venant indeed in various writings intended for practical men treated non-isotropy in considerable detail, but we know of no recent English practical writer who has followed this example. It is very possibly undesirable that elementary books should go into much detail in this matter, but greater prominence should be given to the limitations involved by the assumption of isotropy, whose occurrence is after all probably the exception rather than the rule.

Mathematical treatment.—The author puts *stresses* in the forefront. Supposing the three principal stress directions to coincide with the rectangular axes, then the strain "e_x is made up partly of the direct strain which (the stress) p_x produces in its own direction, and partly of the lateral strains produced by p and p_z" (p. 21). In this way Young's modulus and Poisson's ratio at once come to the front as elastic constants. The general properties of strain, the relations of strain to displacement, the surface stress equations and the ordinary body stress differential equations are not considered. When the basis provided proves too narrow, as it sometimes does—*e. g.*, in Arts. 143 and 147—, the author makes such supplementary assumptions as he finds necessary. This seems a trifle hard on the conscientious student whose knowledge of elasticity is confined to this book. The proof of the "equality of shearing stress in two directions" in Art. 12 is not rigid; the author makes no use of his hypothesis that the cube is indefinitely small, and in fact does not refer at all to bodily forces. The use of $1/\sigma$ for Poisson's ratio may lead to confusion, as that quantity is represented by σ in Thomson and Tait's Natural Philosophy and in Love's Treatise on Elasticity. Again in Art. 147, $1/\sigma$ is replaced by μ—used to represent the rigidity by St. Venant, Pearson, Love, etc.—the rigidity itself being denoted by C. In discussing the determination of the rigidity in Article 72, μ is used for the twisting couple for unit angle of twist. The difference between static and kinetic moduli is presumably an unnecessary refinement in engineering, so that the absence of reference to the subject in Arts. 71 and 72 is hardly surprising, but in Art. 73 the discussion of Maxwell's method of determining the rigidity might well have called attention to the fact that it keeps the tension of the wire unaltered and eliminates the moments of inertia of the shifted masses round their own axes of figure.

Before entering on a discussion of beam problems in Chap. V, the author describes "uniformly distributed stress," which means stress the same in intensity and direction at every point of a plane surface, and "uniformly varying stress" in which the stresses are parallel, but vary as the distance from some line (a "neutral axis") in the plane to which they are perpendicular. He adds (Art. 79): "Uniformly varying stress is practically important because it occurs * * * in a bent beam, in a tie-rod when subjected to a non-axial pull, and in a long strut or column." In short the author *assumes* the ordinary Bernoulli-Euler theory for beams whether locally or generally loaded. He considers first the stresses alone, explaining several graphic methods. He then proceeds to determine the "slopes" and "deflections" in beams, on the assumption that "The strain on any imaginary filament taken along the length of the beam is sensibly the same as if that filament were directly compressed or extended by itself" (Art 97). Supposing x measured parallel to the length of the beam, and denoting the slope by i, the deflection by u, and the radius of curvature of the central axis by R, the author (Art. 99) deduces $1/R = di/dx = d^2u/dx^2$, whence $i = \int(1/R)dx$, $u = \int idx$. Having expressed $1/R$ in terms of the bending couple by the Bernoulli-Euler method, he thus arrives by one step at i and by a second step at u. In introducing the formula $1/R = d^2u/dx^2$ the author specifies that the deflection is assumed very small, but he does not explain that this means that $(du/dx)^2$ is neglected as compared to unity.

The strained form of a bent beam is illustrated by Fig. 84, Art. 102. This would have been clearer if the excess of width at the top had been more exaggerated (cf. Thomson and Tait's Natural Philosophy, Art. 716). Also when the author says: "The lateral strain being $1/\sigma$ of the longitudinal strain, the anticlastic or transverse curvature * * * is $1/\sigma$ of the longitudinal," one cannot but fear that the student will obtain an erroneous idea both of what is meant by "anticlastic curvature" and of what actually takes place. The conclusions in the end of Art. 102 as to the bending of a "wide flat strip" seem hardly in accordance with Pearson and Filon's results (*Quarterly Journal of Pure and Applied Mathematics*, Vol. 31, p. 66, especially Arts. 52 et seq.). The author's treatment of the ordinary first approximation results for beams has much to commend it, especially for students weak at mathematical analysis. When, however, he tries to improve on the first approximation results, he is less satisfactory. Thus in Art. 95 when deducing the distribution of shearing stress over the section of a beam, and in Art. 105, when applying the results so obtained to the calculation of the "additional deflection due to shearing," he tacitly assumes that the shearing stress is the same at every point of a narrow strip perpendicular to the plane of bending. We are not aware that this has been *proved* in any particular case; it is not in accordance with the solution obtained by St. Venant for beams under terminal shearing force (Todhunter and Pearson's History, Vol. 2, Part I, p. 64), or with that found by Pearson and Filon for a heavy elliptic beam (l. c., pp. 88 and 98). Further the "correction" in Art. 105 is of the same order of magnitude as the *difference* between the vertical displacements at different points of the cross section; it was obtained by equating work done by stresses to work done by gravity on the load, while no account was taken of what part of the central section the load was applied at.

In dealing with such matters, a reviewer cannot but regret the great scarcity of references in the work to mathematical text books or papers. There are two or three references to Lord Kelvin and one to Pearson, in connection with experimental work or definitions, while St. Venant is mentioned generally in connection with torsion; but to the text books of these authors, or to that of Love, or to their mathematical papers, there seems no reference anywhere. Thus when the author reaches an unusual result one is uncertain whether it has or has not any more substantial basis than the proof presented in the book. The average engineering student is perhaps unlikely to utilize references, but their presence would at least not harm him and should facilitate enlargement of ideas amongst the more competent.

The treatment of "frames" in Chap. VIII is condensed, and various assumptions,—e. g., in Arts. 113, 116, 117, and

119—seem to require more explanation. The *index* tells us that the "method of reciprocal figures" in Arts. 114 et seq. is due to Maxwell, but his name does not appear in the text. The subject of "redundant members" in frames is very lightly touched upon, and there is no reference to Macaulay's recent extension of Maxwell's method (*Philosophical Magazine*, Jan., 1898, p. 42).

The treatment of struts in Chap. IX is fairly full, but the explanation of Euler's method in Art. 122 might be improved. On p. 172 the curvature is taken as d^2u/dy^2 without restriction. The solution of course assumes that $(du/dy)^2$ is negligible, and yet the conclusion is drawn: "the same force will serve to keep the strut bent whether the curvature is small or not so small." Presumably in engineering practice anything but small curvature is rare, but considerable elastic bending is easily introduced in thin rods, and a reference to some more general treatment (*e. g.*, Love's Elasticity, Vol. 2, Arts. 227 et seq.) would appear desirable. The conclusion reached by the author (Art. 122) that the length of a segment of a bent strut may be any submultiple, even or odd, of the whole length, is correct, but it does not follow from his mathematics. His final equation justifies his conclusion $(P/EI)^{1/2}(L/2) = n\pi/2$, only when n is an *odd* integer. The fact is that in originally assuming $u = u_1 \cos\{(P/EI)^{1/2}y\}$ the author excluded all cases in which the displacement u vanishes at the center of the rod (his origin). The illustration selected in Art. 126 of "struts with lateral load" is rather curious. A bending moment whose true value is $(WL/8)\{1 - (2x/L)^2\}$ is represented as $(WL/8)\cos(\pi x/L)$, thus replacing the true differential equation by one of the type

$$(d^2u/dx^2) + au + c \cos bx = 0.$$

A pure mathematician, it may be feared, would hardly accept as adequate the explanations that "a parabola * * * is not far from coincidence with a curve of sines" and that $1 - (2x/L)^2$ agrees in value with $\cos(\pi x/L)$ both when $x = 0$ and when $x = L/2$. It may be added—as a personal contribution from the reviewer—that while the solution of the *true* differential equation presents no real difficulty, the result does not lend itself to manipulation by quite elementary methods.

Chap. X calls for a few criticisms. Art. 127 assumes—as is indeed usual—that when a circular shaft of any length is twisted in any symmetrical way each element suffers a displacement directly proportional to its distance from the axis. There are in reality any number of possible symmetrical kinds of twisting, the species depending on the force system applied, and unless the length be great compared to the diameter there is no *a priori* reason to suppose that the stress and strain will even approximate to the ordinary simple law (see *Cambridge Transactions*, vol. 14, p. 365). The crank shaft dealt with in Art. 131 is a case where the application of the Bernoulli-Euler method is somewhat speculative. The author seems here to hesitate between the greatest principal stress and the maximum shearing stress as the criterion for rupture, but his language is rather ambiguous. In Art. 133, dealing with torsion beyond the elastic limit, the author supposes that the twisting couple can be increased until practically the whole of the material—starting with the outer layers—passes beyond the elastic limit, and that the altered material possesses everywhere the same shearing stress q'; he speaks of q' as "the ultimate shearing stress of the material." In Arts. 84 and 85 the author presented a closely similar view in dealing with the bending of rectangular beams; stating that the stress may approach a limiting condition in which there is a uniform longitudinal tension over one half the cross section and an equal compression over the other half. It would be desirable that the conditions should be clearly stated under which material may be expected to behave in the way assumed in these articles, and that there should be references to experiments on the subject. The point seems of importance in connection with theories of rupture. In Arts. 134 and 135 dealing with spiral springs a reference to some more complete treatment of the subject would be particularly desirable. When the author (p. 197) talks of resolving "angles about the axes ox and oy to find the horizontal and vertical components of the angular displacement * * *" his meaning may be plain to the practical man, but it certainly is not so to one habituated to a different usage of mathematical terms. Art. 138 dealing with the whirling of shafts is more satisfactory, but it might be desirable to tell the student that the formala $d^2M/dx^2 = 4\pi^2n^2wu/g$, which is really assumed in the proof, is obtainable by ordinary statics. It would also be well to refer to the possibility that the material may be strained beyond the elastic limit before "whirling" can come in (*Proceedings of the Cambridge Philosophical Society*, vol. 7, p. 283).

Chapter XI deals with several interesting problems, but the author is handicapped, as in Chapter X, by his avoidance of the less elementary methods. The assumption that "hoop stresses" are the same at every point of the thickness of a thin shell—whether spherical, or cylindrical (of circular or elliptical section)—is a short way of reaching results, but it has the serious disadvantage of possessing limitations on which it throws no light. In the "oval section," Article 142, the hypothesis seems of doubtful validity under any conditions (cf. Todhunter and Pearson's History, Vol. 2, Part 1, Arts. 537–538). Articles 143–4 treating of the thick circular cylinder are less elementary and more satisfactory; but the author has to introduce the supplementary assumption that the longitudinal strain is uniform, and he commits himself to the greatest stress theory of rupture. As this is a specially good example of the difficulty of getting along without some general theory of rupture, and also of the expediency of knowing what the true theory is, the reviewer ventures on a slight addition to the text. The displacements u along the radius from the cylindrical axis, and w parallel to its length taken as the z-axis, are of the type (*Proceedings of the Cambridge Philosophical Society*, Vol. 7, p. 204)

$$u = ar + D/r, \qquad w = Az,$$

where A, a, D are constants determined from the surface conditions. Suppose these conditions to be uniform pressures p_i and p_e over the inner and outer cylindrical surfaces, whose radii are r_1 and r_2, along with uniform tension P parallel to the cylindrical axis (we may suppose P to represent the action of closed ends when such exists). Then denoting Young's modulus by E and Poisson's ratio by η, we easily find

$$u = r\left\{\frac{1-\eta}{E}\,\frac{r_1^2 p_i - r_2^2 p_e}{r_2^2 - r_1^2} - \eta\,\frac{P}{E}\right\} + \frac{1+\eta}{E}\,\frac{r_1^2 r_2^2}{r}\,\frac{p_i - p_e}{r_2^2 - r_1^2};$$

$$w = z\left\{\frac{P}{E} - \frac{2\eta}{E}\,\frac{r_1^2 p_i - r_2^2 p_e}{r_2^2 - r_1^2}\right\}.$$

Art. 143 treats the case when P and p_e vanish, for which

$$u = \frac{p_i}{E}\,\frac{r_1^2}{r_2^2 - r_1^2}\left\{(1-\eta)\,r + (1+\eta)\,r_2^2/r\right\};$$

$$w = -\frac{2\eta p_i}{E}\,\frac{r_1^2 z}{r_2^2 - r_1^2}.$$

The corresponding principal stresses, in Todhunter and Pearson's notation (the radial, the "hoop," and the longitudinal) are

$$\widehat{rr} = -p_1(r_1/r)^2(r_2^2 - r^2)/(r_2^2 - r_1^2);$$

$$\widehat{\varphi\varphi} = p_1 r_1^2(1 + r_2^2/r^2)/(r_2^2 - r_1^2); \quad \widehat{zz} = 0.$$

Everywhere $\widehat{\varphi\varphi}$ is the greatest stress, $\widehat{\varphi\varphi} - \widehat{rr}$ the maximum stress-difference, and u/r the greatest strain; and all three quantities have their greatest values over the inner cylindrical surface, where

$$\widehat{\varphi\varphi} = p_1(r_1^2 + r_2^2)/(r_2^2 - r_1^2); \quad \widehat{\varphi\varphi} - \widehat{rr} = 2p_1 r_2^2/(r_2^2 - r_1^2);$$

$$u/r = (p_1/E)\{r_2^2 + r_1^2 + \eta(r_2^2 - r_1^2)\}/(r_2^2 - r_1^2).$$

Now suppose, as in Art. 143, that f is the largest safe tension in a long bar of the material; then f is a superior limit: on the greatest stress theory to $\widehat{\varphi\varphi}$, on the maximum stress difference theory to $\widehat{\varphi\varphi} - \widehat{rr}$, and on the greatest strain theory to $E(u/r)$. Thus if the limiting values allowed by the three theories to the internal pressure be respectively p_1', p_1'' and p_1''', we have

$$p_1' = f(r_2^2 - r_1^2)/(r_2^2 + r_1^2); \quad p_1'' = f(r_2^2 - r_1^2)/(2r_2^2);$$

$$p_1''' = f(r_2^2 - r_1^2)/\{r_2^2 + r_1^2 + \eta(r_2^2 - r_1^2)\};$$

and so
$$p_1'' = p_1'(r_2^2 + r_1^2)/(2r_2^2);$$

$$p_1''' = p_1' \div \{1 + \eta(r_2^2 - r_1)/(r_2^2 + r_1^2)\}.$$

As $r_2^2 > r_1^2$, clearly p_1' is always larger than p_1'' or p_1'''; or the greatest stress theory—of which alone our author takes cognizance—allows a greater internal pressure than either of the other two theories would consider safe. In a very thin shell p_1', p_1'', and p_1''' approach equality; but in a very thick shell we have ultimately $p_1'' = p_1'/2$, $p_1''' = p_1'/(1 + \eta)$.

"The use of initial internal stress in strengthening a thick tube" (Art. 145) represents a case where a good deal may depend on the theory of rupture adopted. Art. 146 assumes the stress in a revolving ring to be uniform over the cross section. Presumably this is approximately true when the diameter of the cross section is small compared to the aperture of the ring, but not otherwise, and the conclusions arrived at should be correspondingly restricted.

Art. 147 dealing with the "stress in a revolving disc" is perhaps the nearest approach in the book to the use of the strict mathematical theory. In drawing conclusions as to the fastest safe speed of rotation the author again confines himself to the greatest stress theory. It happens, however, that all three rupture theories mentioned above allow the same limiting velocity in the case of a disc with a hole. In the disc without a hole the greatest stress and maximum stress difference theories likewise agree, but the greatest strain theory allows a velocity greater than that allowed by the other two theories in the ratio $1 : \sqrt{1 - \eta}$. The problem of the rotating disc has had a somewhat curious history, an account of which—with a more complete treatment of the problem—will be found in the *Proceedings of the Cambridge Philosophical Society*, Vol. 7, p. 201.

Units.—Those who regard the retention of British units as a sign post on the road to destruction will regard the author as situated far on the downward track. There seems no reference to the C. G. S. system except in footnotes on pp. 3 and 53, which give the relation between tons or lbs. per square inch and kilogrammes per sq. cm. The tables of strength and of elastic moduli at the end use "tons per sq. inch," whilst the table of the "approximate weights of materials" uses "lbs. per cubic foot." In Art. 130 work is measured in "inch-pounds" per minute. In the text the author speaks of tons or lbs. per sq. inch (not ton *weights* etc.), and his attitude towards "g" when he has to use it is apt to be reserved (see Art. 72). Our own view is that so long as it is sufficiently clear what is meant, the terminology employed in referring to the unit of stress is of secondary importance. So long as "factors of safety" are as elastic (in the usual sense) as at present, it does not much matter whether the standard ton resides in London or New York.

This review has been written under the belief that what the readers of the BULLETIN are accustomed to is a critical examination of books, and not the kind of general appreciation that is so easy to write and so pleasant (for the author's uncritical friends) to read. Lest, however, the BULLETIN may occasionally fall into the hands of uncritical persons, it may be well to state explicitly that in many respects Professor Ewing's book is an excellent one, especially as an introduction to the subject. In general, it is clearly written and well arranged, and bears evidence of the author's experience as a teacher. My criticisms largely arise from the fact that the author seldom indicates the limitations and uncertainties in the mathematical methods adopted, or supplies references whence his readers might obtain the knowledge which the book does not itself supply. Even in the parts of the work dealing with experimental results one is rather struck by the paucity of references to recent work done outside the author's laboratory. The insertion of long lists of authorities in an elementary book may savor of ostentation, but the author goes too far in the opposite direction.

The book seems carefully printed. Of the few errata I have noticed the following are the chief not already referred to :—p. 84, l.19, for "then" read "there"; p. 101, last line, for "negative" read "positive"; p. 108, lines 22 and 23, interchange "upper" and "lower" (?); p. 111, l. 6 from foot, insert "greatest" before "stress"; p. 117, case 4, the force F_s *increases* uniformly (*algebraically*) from one end to the other; p. 126, the conclusion that shearing stress is greatest at the neutral axis would follow from proof given only if breadth ζ constant; p. 135, in lines 9 and 10 of Art. 102, interchange "above" and "below"; p. 158, l. 2 below fig. 101, for 188 read 116; p. 185, l. 3 from foot, for "M" read "M_1"; p. 207, "t" is omitted in denominators of formulæ for f and f' in Art. 142; p. 208, "θ" should be shown in Fig. 136; p. 227, l. 10, for dM/da read dM_p/da.

CHARLES CHREE.

RICHMOND, SURREY,
July 2, 1900.

SCHEFFERS' DIFFERENTIAL GEOMETRY.

Anwendung der Differential- und Integralrechnung auf Geometrie. By DR. GEORG SCHEFFERS, Professor in the Darmstadt Polytechnic School. Erster Band: *Einführung in die Theorie der Curven in der Ebene und im Raum.* Leipzig, Veit and Co., 1901.

THE author of this work has already proved his capacity for writing text-books in a clear and readable manner, in the three volumes of Lie's works which he edited. The present volume is arranged and written in the same attractive and on the whole satisfactory style—for which the author is to be particularly congràtulated, inasmuch as the work is intended for students who wish to obtain a bird's-eye view of the vast subject of the applications of the infinitesimal calculus to geometry—a subject which is touched upon, more or less, by nearly every writer of mathematical text-books or papers, and which for that reason is almost inaccessible as a whole to the average student.

There is a remarkable dearth of books on this subject in the English language—practically the only sources of information for the beginner being Salmon's classical (but antiquated) works, and the various larger text-books on the calculus. Important portions of the subject are treated by all the French authors who edit a Cours d'analyse—Hôuël, Jordan, Picard, Laurent, etc.; but these works fail to be adapted to the wants of the beginner either from being too much condensed—as those of Jordan and Picard—or from being too diffuse, as that of Laurent. In German, the elementary text-books are Hoppe and Joachimsthal—neither of which appeals to the beginner on account of peculiarities of arrangement; while Stahl-Kommerell is much too brief and Bianchi too difficult for one who has only had a course in elementary calculus.

A great mathematical discovery is nearly always the outgrowth of a need, felt by mathematicians in general rather than by a single investigator, for the advance represented by the discovery; so that a work which coördinates and presents in a readable form the principal branches of a many-sided subject paves the way, at least, for a notable advance in the science. In this sense, Professor Scheffers deserves the thanks, both of the students of mathematics, for whom the work is especially intended, and of mathematicians in general.

The volume which has just appeared, and which we shall examine briefly, contains the first grand division of the author's subject: the theory of curves—plane curves, and space curves with their accompanying developables. The second division, which will appear within a year, will contain the general theory of surfaces.

The subjects considered in Part I. of the present volume are largely those given in any complete work on the calculus—the theory of contact of plane curves, curvature, evolute and involute, etc. It is perhaps regrettable that practically nothing is done towards the discussion of asymptotes (as is done, for example, in Houël's Calcül infinitésimal, volume 2), and the treatment of singular points is very meager.

Some sections of this part, however, treat of subjects not usually given in the current text-books, and they deserve special notice. In § 2 the writer explains what is meant by a *movement* in the plane, by showing that it is immaterial whether the operations represented by the familiar equations

$$\begin{aligned} x' &= x \cos \alpha - y \sin \alpha + a, \\ y' &= x \sin \alpha + y \cos \alpha + b, \end{aligned} \qquad (1)$$

be interpreted, 1°, as a change in the position of the coordinate axes with reference to the geometrical figure under consideration, or 2°, as a change of the position of the figure with reference to the (unchanged) axes. This introduces the reader at once, and in the most natural manner, to the conception of a *transformation* of the points of the plane. In § 8 certain *differential invariants* of a plane curve are defined and deduced by means of the definition of a movement contained in § 2. Any function of

$$x, \quad y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{d^2x}, \cdots$$

which remains unchanged, when the curve $y = f(x)$ is subjected to the movements (1), is called a differential invariant of the curve with regard to those movements.* All possible differential invariants (in this sense) are shown—in an entirely elementary manner and without reference to the theory of transformation groups—to be functions of the magnitudes

$$r, \quad \frac{dr}{d\tau}, \quad \frac{d^2r}{d^2\tau}, \cdots, \qquad (2)$$

where r and τ are radius of curvature and angle of contingence, respectively.

The results of this section are applied in § 9 to establish the necessary and sufficient conditions that two plane curves shall be congruent. For since the differential invariants (2) are unchanged by a movement (1), it is clear that two curves can be congruent only when their differential invariants have the *same* values at corresponding points. It is clear that $dr/d\tau$ is some function of τ; so that if $dr/d\tau = \omega(\tau)$ along the one curve, the necessary and sufficient condition for congruence may be seen to be that $dr/d\tau$ must be the *same* function of r along the other curve.*

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* The conception "differential invariant" is contained implicitly in all the older works on differential equations. Cayley, in his Theory of Invariants, and Halphen, in his "Thèse sur les invariants différentiels" (1878), established certain classes of differential invariants; but the *general* theory of differential invariants is due to Lie (*Verhand. der Gesell. der Wiss. zu Christiania*, Febr., 1875, also 1882–3; *Archiv for Math.*, 1882–3; *Math. Annalen*, vol. 24).

All properties of a given curve therefore which are independent of the position of the coördinate axes, find their complete expression in the equation $dr/d\tau = \omega(\tau)$; therefore nothing is more natural than to introduce $dr/d\tau = \omega(\tau)$ as the intrinsic, or natural, equation to a curve.†

In § 11 it is shown that the differential invariants (2) represent, geometrically, the radii of curvature of the evolute, or of the evolute of the evolute, ···, of the given curve, as readily appears.

The §§ 14–20 of Part I. are taken up with a brief outline of the theory of ordinary differential equations in the plane: and the introduction of the conceptions *curvilinear coördinates*, *curve-nets*, etc. It is very fortunate for the beginner that the author treats the subject of parametric (curvilinear) coördinates in the plane very fully and clearly—carefully deducing the conditions that two curve families shall form an orthogonal system, an isometric system, etc., in the plane—as most of these results can be extended at once to curve families on a surface. It is just these sections of Part I. which will be most useful and most interesting to the beginner—as those subjects are not systematically treated in the current text-books, at least until the difficult general theory of surfaces is studied. Of especial interest are the paragraphs giving Lie's geometrical interpretation of an integrating factor, the finding of all transformations which leave areas invariant, the finding of all *conform* transformations, and the discussion of Lie's method of integrating *two* ordinary differential equations when a certain relation between their integrating factors is known.

Part II. of this volume scarcely demands a detailed discussion here, as it consists mainly of an excellent exposition of the classical theory of the space-curve, introducing the reader to the conceptions *osculating plane*, *contact*, *curvature*, *osculating sphere*, etc., together with the natural extension to space of the developments in Part I. concerning movements, differential invariants, etc.

In § 3, for instance, the writer explains how the (space) equations analogous to (1) define a movement, and proves incidentally Mozzi's theorem that (in general) any movement in space can be replaced by a screw movement.

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* This criterion does not hold, for obvious reasons, for circles and minimal straight lines.

† That this is equivalent to Whewell's form $s = f(\tau)$ is readily verified.

Then in § 12 all differential invariants of a space curve—with regard to the movements—are determined; and they are shown to be functions of the radius of curvature r, the radius of torsion ρ, and the derivatives of these functions with respect to the arc length s. Thence are obtained the intrinsic (or natural) equations of space curves as the two independent equations which contain the differential invariants of lowest orders of the curve. In the next section the author gives Lie's method of reducing to the integration of a Riccati equation the problem of finding the finite equations of a space curve from its given intrinsic equations.

These sections, as well as § 18, 19 in which the curve is to be determined from the spherical indicatrix of its tangents, principal normals, or binormals, will offer considerable, but by no means insuperable, difficulties to the beginner.

In order to complete the development of the theory of the space curve—in particular, the theory of the evolutes or involutes of a given space curve—it is necessary to examine the curve in connection with its accompanying developable surfaces. The more important of these developables are, respectively, the envelopes of the osculating planes and of the normal planes of the curve. The author gives, therefore, in the first section of Part III. a brief discussion of the ruled surface, and, in particular, of the developable, deducing afterwards the theory of the evolutes and involutes of the space curve in his usual clear manner.

All theorems which are proved with the arc length s chosen as the variable parameter of which x, y, and z are functions, fail of course for the special curves for which s is everywhere zero. Hence the minimal curves (or minimal straight lines) demand a special discussion, and the development of the theory of these curves according to Legendre, Enneper, Weierstrass, and Lie closes the volume.

The type work of the book is excellent; very few misprints have been observed. It is furnished with an appendix containing several tables of useful formulæ, and with a good index. The author has enriched the general theory of curves by several original developments—notably in the discussion of the trajectories of a curve family * in the plane, and in the integration of the intrinsic equations † of a space curve. The historical references, given whenever a new definition is introduced, will be of great value to the

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* G. Scheffers, *Ber. der math.-phys. Klasse der kgl. Sächs. Gesell. der Wiss. zu Leipzig*, October 24, 1898.

† Same publication, January 8, 1900.

reader who wishes to penetrate deeper into this branch of the science. The figures in the text are drawn with great care, and the illustrative examples worked out fully and clearly. The reader of English text-books will miss the long lists of problems usually given at the ends of the chapters in such works; but we predict for the book a very useful career in the lecture room, especially in the hands of an energetic teacher who can supply himself with an abundance of illustrative problems.

J. M. PAGE.

UNIVERSITY OF VIRGINIA,
November 11, 1900.

NOTES.

A NEW edition of the Annual Register of the Society will be issued in January. Forms for furnishing necessary information have been sent to each member, and a prompt response will be of great assistance to the Secretary.

AT the annual general meeting of the London mathematical society held November 8, 1900, the following officers were elected: Dr. E. W. HOBSON, president; Lord KELVIN, Professor W. BURNSIDE, and Major P. A. MACMAHON, vice-presidents; Dr. J. LARMOR, treasurer; Mr. R. TUCKER, and Professor A. E. H. LOVE, honorary secretaries; Mr. J. E. CAMPBELL, Lieut.-Col. A. C. CUNNINGHAM, Professor E. B. ELLIOTT, Dr. J. W. L. GLAISHER, Professor M. J. M. HILL, Messrs. A. B. KEMPE, H. M. MACDONALD, A. E. WESTERN and E. T. WHITTAKER, additional members of the council. The subject of LORD KELVIN's address as retiring president of the society was "The transmission of force through a solid."

AT the anniversary meeting of the Royal Society of London, on November 30, Sir WILLIAM HUGGINS was elected president. Among the new members of the council is Professor E. B. ELLIOTT. A Royal medal was presented to Major P. A. MACMAHON for his contribution to mathematical science.

THE National academy of sciences held its autumn meeting at Brown University, Providence, R. I., on November

13. The following mathematical and physical papers were among those read:

J. TROWBRIDGE: "Investigations of light and electricity with the aid of a battery of twenty thousand cells."

S. L. PENFIELD: "Stereographic projection and some of its possibilities from a graphical standpoint."

T. C. MENDENHALL: "Note on the energy of recent earthquakes."

H. A. ROWLAND: "On the explanation of inertia and gravitation by means of electrical phenomena."

C. BARUS: "On stability of vibration and on vanishing resonance."

THE Royal Academy of Belgium proposes the following prize questions for the year 1901:

1. Find the form of the principal terms introduced by the elasticity of the earth's shell into the formulas for the change of obliquity and of longitude. Value of the prize, 800 francs.

2. An important contribution is to be made to the theory of mixed forms in any number of series of variables, and the results are to be applied to the geometry of the corresponding space. Value of the prize, 600 francs.

The memoirs may be written in French or Flemish, and should be sent before August 1, 1901, under the usual conditions as to anonymity to the permanent secretary of the academy in Brussels.

UNIVERSITY OF PARIS. The following courses in mathematics are among the announcements of the faculty of sciences for the academic year 1900–1901:—First semester:—By Professor G. DARBOUX: Fundamental principles of infinitesimal geometry especially with reference to the theory of triple systems of orthogonal surfaces, two hours.—By Professor E. GOURSAT: The operations of the differential and integral calculus, and analytic functions, two hours.—By Professor P. APPELL: The general laws of equilibrium and motion, two hours.—By Professor H. POINCARÉ: The motions of celestial bodies about their centers of gravity, two hours.—By Professor BOUSSINESQ: On the internal friction of fluids, two hours.—By Professor G. KŒNIGS: Kinematics of solid and deformable bodies, with application to the study of machines, two hours.—By Professor L. RAFFY: Elements of analysis and mechanics, two hours.—By Dr. M. ANDOYER: General theory of planetary perturbations, one hour.—Conferences by Professors RAFFY, HADAMARD, and PUISEUX, and Messrs. ANDOYER and BLUTEL.—The preliminary announcements for the second semester include courses by Professor E. PICARD on algebraic functions, by Professor E. GOURSAT on differential equations, by Professor P. APPELL on analytical mechanics, by Professor G. KŒNIGS on the study of machines, and by Professor L. RAFFY on the differential equations of mechanics and physics.

OXFORD UNIVERSITY. Advanced mathematical courses for the Michaelmas term of the current academic year are announced as follows:—By Professor E. B. ELLIOTT: Theory of numbers, two hours; substitutions and resolvents, one hour.—By Professor W. ESSON: Analytic geometry of plane curves, two hours; synthetic geometry of plane curves, one hour.—By Professor A. E. H. LOVE: Gravitational attraction and theory of the potential, three hours; theory of sound, one hour.—By Mr. J. E. CAMPBELL: Differential equations, two hours.—By Mr. H. T. GERRANS: Three-dimensional rigid dynamics, two hours.

CAMBRIDGE UNIVERSITY. Mr. A. N. WHITEHEAD and Mr. A. BERRY are the examiners, and Mr. J. G. LEATHAM and Mr. J. H. GRACE the moderators for Part I. of the mathematical tripos, 1901. Professor H. LAMB, Mr. H. W. RICHMOND, Mr. H. F. BAKER and Mr. H. M. MACDONALD are the examiners for Part II. of the same tripos.

THE Smith's prizes for 1900 have been granted to J. F. CAMERON for his essay: "The molecules as electric oscillators," and to R. W. H. T. HUDSON for "Differential equations of the second order and their singular solutions." Mr. Hudson was senior wrangler and Mr. Cameron second wrangler in 1898.

THE library of the late DR. M. C. VERLOREN VAN THEMAAT, containing a number of rare mathematical works, will be sold at auction by Frederick Muller and Company, Doelenstraat 10, Amsterdam, Holland, on December 18–19.

IN a pamphlet of forty-seven pages, Ueber die Nomographie von M. d'Ocagne, Leipzig, Teubner, 1900, Professor F. SCHILLING presents the salient ideas of M. d'Ocagne's book (reviewed in the BULLETIN, volume 5, p. 362) under the conviction of its great utility for all branches of applied mathematics. As this utility seems generally admitted, we may hope that a series of abaci for practical use will soon be accessible.

According to *Nature*, Dr. Cullis, professor of mathematics, at Hartley College, Southampton, has been appointed professor of mathematics at the Presidency College, Calcutta; and Mr. J. F. Hudson, late lecturer in mathematics at Jesus College, Oxford, has been appointed professor of mathematics at Hartley College, Southampton.

Professor Frank Morley of Johns Hopkins University has recently been appointed editor of the *American Journal of Mathematics*.

Professor A. Tresse has been appointed professor of mathematics at the collège Rollin, Paris.

Dr. H. Lorenz, of Halle, has been made director of the physical and technical institute of the university of Göttingen.

Professor G. Oltramare, the venerable dean of the faculty of sciences of the university of Geneva, has retired from the chair of higher mathematics at the age of eighty-four, after a tenure of fifty-two years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bonola (R.). Sulla teoria delle parallele e sulla geometria non-Euclidea. Bologna, 1900. 8vo. 80 pp., 2 plates.

Buroni (S.). See Lavaggi (L.).

Cantor (M.). Vorlesungen uber Geschichte der Mathematik. (In 3 Banden.) Vol. III: Von 1668–1758. Abteilung 1: Von 1668–1699. 2te Auflage. Leipzig, Teubner, 1900. 8vo. 261 pp. M. 6.60

Ermakof (V. P.). Lectures on the integral calculus. (Russian.) Parts 1 and 2. Kief, 1900. 8vo. 350 pp. R. 3.00

Frenet (F.). Collection of problems in the infinitesimal calculus. Part 2: Integral calculus; part 3: Miscellaneous problems and appendix. Translated into Russian from the 5th French edition by A. P. Nenashef. (Russian.) Moscow, 1900. 8vo. 4 + 240 pp. R. 2.50

FRIEDRICH (M.). Katechismus der analytischen Geometrie. 2te Auflage, durchgesehen und verbessert von E. Riedel. Leipzig, Weber, 1900. 12mo. 8 + 217 pp. (Weber's illustrirte Katechismen, No. 116.) Cloth. M. 3.00

HAGEN (J. G.). Synopsis der höheren Mathematik. Vol. 3. Lieferung 2. Berlin, Dames, 1900. 4to. M. 5.00

HENKE (R.). See SCHLÖMILCH (O.).

JAHRBUCH über die Fortschritte der Mathematik. Begrundet von C. Ohrtmann. Unter besonderer Mitwirkung von F. Muller, A. Wangerin und G. Wallenberg herausgegeben von E. Lampe. Vol. 29: Jahrgang 1898. (In 3 Heften.) Heft 2. Berlin, Reimer, 1900. 8vo. Pp. 417–576. M. 5.00

LAMPE (E.). See JAHRBUCH.

LAVAGGI (L.). Calcolo infinitesimale. Lezioni [dettate nell' anno] 1899–1900 nella r. università di Parma, compilate per cura di S. Buroni. Disp. 71–102 (ultima). Parma, Zafferri, 1900. 8vo. Pp. 561–815.

LÜBECK (O.). Algebraische Analysis. Unterweisungen und Beispiele. Strelitz, Hittenkofer, 1900. 8vo. 30 pp. (Unterrichtswerke fur Selbstunterricht und Bureaugebrauch, Methode Hittenkofer.) M. 2.00

MÜLLER (F.). Mathematisches Vokabularium französisch-deutsch und deutsch-franzosisch, enthaltend die Kunstausdrücke aus der reinen und angewandten Mathematik. 1ste Halfte. Leipzig, Teubner, 1900. 8vo. 9 + 132 pp. M. 8.00

——. See JAHRBUCH.

NENASHEF (A. P.). See FRENET (F.).

PENSA (A.). Sull' influenza di alcune singolarità di superficie sul genere numerico e sul bigenere P, con applicazioni alla determinazione di superficie razionali di quinto ordine. Mondovi, Vescovile, 1900. 8vo. 28 pp.

PIZZARELLO (D.). Sulle funzioni trascendenti intere; memoria presentata come tesi di laurea nella r. università di Roma il 14 novembre, 1899. Messina, Tipografia dell' *Epoca*, 1900. 8vo. 70 pp.

RIEDEL (E.). See FRIEDRICH (M.).

SCHLÖMILCH (O.). Uebungsbuch zum Studium der höheren Analysis. Teil II: Aufgaben aus der Integralrechnung. 4te Auflage, bearbeitet von R. Henke. Leipzig, Teubner, 1900. 8vo. 8 + 448 pp. M. 9 00

WALLENBERG (G.), WANGERIN (A.). See JAHRBUCH.

WILCZYNSKI (E. J.). On continuous binary linearoid groups, and the corresponding differential equations and A functions. (*American Journal of Mathematics*, Vol. 22, pp. 191–225.) 4to.

——. An application of group theory to hydrodynamics. (*Transactions of the American Mathematical Society*, Vol. 1, pp. 339–352.) 4to.

——. Poetry and mathematics. Berkeley, University Press, 1900. 8vo. 16 pp.

II. ELEMENTARY MATHEMATICS.

Baroni (E.). Sui metodi elementari per la risoluzione dei problemi geometrici. Bologna, Zanichelli, 1900. 8vo. 23 pp.

Berry (T. W.) and Vine (G. F.). How to work deductions in Euclid. London, Simpkin, 1900. 12mo. 112 pp. 2 s.

Dupuis (N. F.). Principles of elementary algebra. Revised and corrected edition. New York, Macmillan, 1900. 12mo. 7 + 336 pp. Cloth. $1.10

Durfee (W. P.). Elements of plane trigonometry. Boston, Ginn, 1900. 16mo. 6 + 105 pp. Cloth. $0.80

Faudella e Pantaleo. Aritmetica e geometria per la quarta classe elementare. Torino, Paravia, 1900. 16mo. 128 pp. Fr. 0.60

——. Aritmetica e geometria per la quinta classe elementare. Torino, Paravia, 1900. 16mo. 99 pp. Fr. 0.60

Gazzaniga (E. P.). Aritmetica generale: divisioni ordinarie, speciali e mobili, configurazioni numeriche e numeri primi, divisioni e frazioni periodiche, nuovi studî e teoremi. Bergamo, Bolis, 1900. 8vo. 4 + 213 pp. Fr. 5.00

Giordano (D.). Nozioni di aritmetica razionale, esposte per uso del ginnasio superiore. Ragusa inferiore, Criscione, 1900. 16mo. 102 pp. Fr. 1.00

Holzmann (A.) und Massinger (R.). Geometrische Anschauungslehre (in 3 Teilen) im Anschluss an den Lehrplan der badischen Realschulanstalten. Teil I: Erster Abschnitt der ebenen Gebilde; Pensum der Klasse IV; 32 pp. Teil II: Zweiter Abschnitt der ebenen Gebilde; Pensum der Klasse V; 30 pp. 2te Auflage. Karlsruhe, Reiff, 1900. 8vo. Boards. M. 0.60

Jacquet (E.) et Laclef (A.). Cours d'arithmétique théorique et pratique, avec de nombreux exercices, problèmes, etc., à l'usage des écoles normales d'instituteurs et d'institutrices, des écoles primaires supérieures, des écoles professionnelles et des candidats au brevet supérieur. Paris, Nathan, 1901. 18mo. 8 + 317 pp.

Juling (G.). Fünfstellige Logarithmentafeln für Schüler. Leipzig, Berger, 1900. 8vo. 144 pp. Cloth. M. 1.20

Kutnewsky (M.). See Müller (H.).

Laclef (A.). See Jacquet (E.).

Lichtblau (W.) und Wiese (B.). Sammlung geometrischer Rechenaufgaben, zum Gebrauch an Seminarien, sowie zum Selbstunterricht bestimmt und herausgegeben. 2te Auflage. Breslau, Hirt, 1900. 8vo. 164 pp. M. 1.75

Lieber (H.) und Lühmann (F. von). Leitfaden der Elementar-Mathematik. Neu herausgegeben von C. Musebeck. Teil I: Ausgabe B für Realgymnasien, Oberreal- und Realschulen. Planimetrie. Einfuhrung in die Trigonometrie und Stereometrie. (Lehraufgabe der Quarta bis Untersekunda.) Berlin, Simion, 1900. 8vo. 5 + 109 pp. M. 1.60

Lühmann (F. von). See Lieber (H.).

MASSINGER (R.). See HOLZMANN (A.).

MÜLLER (H.) und KUTNEWSKY (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Ausgabe A, für Gymnasien, Realgymnasien und Oberrealschulen. Teil I. Leipzig, Teubner, 1900. 8vo. 8 + 315 pp. Cloth. M. 2.80

——. Ausgabe B, für Realschulen. Teil I. Leipzig, Teubner, 1900. 8vo. 8 + 289 pp. M. 2.60

MÜSEBECK (C.). See LIEBER (H.).

NUÑEZ (M. G.). Tratado completo de matematicas. Cuaderno 1: Operaciones fundamentales con numeros enteros. Madrid, 1900. 4to. Pp. 1–82. Fr. 3.50

PANTALEO. See FAUDELLA.

PAYNE (E. L.). Principles and methods of college algebra. Emporia, Kan., M'Cord & M'Cord, 1900. 12mo. 56 pp. $0.25

RIEDEL (E.). Katechismus der Planimetrie, mit einem Anhang über harmonische Teilung, Potenzlinien und das Beruhrungsproblem des Apollonius. Leipzig, Weber, 1900. 12mo. 10 + 346 pp. (Weber's illustrirte Katechismen, No. 225.) Cloth. M. 4.00

SERRET (J. A.). Traité de trigonométrie. 8e édition. Paris, Gauthier-Villars, 1900. 8vo. 10 + 336 pp. Fr. 4.00

VINE (G. F.). See BERRY (T. W.).

WIESE (B.). See LICHTBLAU (W.).

III. APPLIED MATHEMATICS.

ADAMCZIK (J.). Compendium der Geodäsie. Wien, Deuticke, 1900. 8vo. 8 + 515 pp. M. 10.00

BURONI (S.). See CARDANI (P.).

CALDARERA (F.). Corso di meccanica razionale. Vol. I: Cinematica. Palermo, 1900. 8vo. 329 pp. Fr. 12.50

CARDANI (P.). Fisica matematica. Lezioni [dettate nell'anno scolastico] 1899–1900 nella r. università di Parma e compilate per cura di S. Buroni. Disp. 31–40 (ultima). Parma, Zafferri, 1900. 8vo. Pp 241–320.

CASTLE (F.). Workshop mathematics. Part I. London and New York, Macmillan, 1900. 12mo. 8 + 154 pp. Cloth. $0.35

——. Part II. London and New York, Macmillan, 1900. 12mo. 9 + 177 pp. Cloth. $0.35

DÉCOMBE (L.). La célérité des ébranlements de l'éther. Evreux, Hérissey, 1900. 16mo. 96 pp. (Scientia, physique mathématique, No 9.)

EMTAGE (W. T. A.). Elementary mechanics of solids. London and New York, Macmillan, 1900. 12mo. 338 pp. $0.60

FÖPPL (A.) Vorlesungen über die technische Mechanik. Vol. 2: Graphische Statik. Leipzig, Teubner, 1900. 8vo. 10 + 452 pp. Cloth. M. 10.00

FUHRMANN (A.). Anwendungen der Infinitesimalrechnung in den Naturwissenschaften, im Hochbau und in der Technik. Lehrbuch und Aufgabensammlung. (In 6 Teilen.) Teil I: Naturwissenschaftliche Anwendungen der Differentialrechnung. 2te Auflage. Berlin, Ernst, 1900. 8vo. 18 + 239 pp. M. 6.00

GUILHAUMON (J. B.). Résumé de théorie du navire. 2e édition. Paris, Berger-Levrault, 1900. 8vo. 84 pp., 2 plates. Fr. 2.50

HERMAN (R. A.). A treatise on geometrical optics. London, Clay, 1900. 8vo. 354 pp. 10s.

HERTZ (H.). Electric waves; researches on the propagation of electric action with finite velocity through space. Authorized English translation by D. E. Jones, with a preface by Lord Kelvin. London, Macmillan, 1900. 8vo. 298 pp. 10s.

KELVIN. See HERTZ (H.).

JONES (D. E.). See HERTZ (H.).

LAUENSTEIN (R.). Die Festigkeitslehre. Elementares Lehrbuch für den Schul- und Selbstunterricht sowie zum Gebrauch in der Praxis, nebst einem Anhang enthaltend Tabellen der Potenzen, Wurzeln, Kreisumfänge und Kreisinhalte. 6te Auflage. Stuttgart, Bergstrasser, 1900. 8vo. 6 + 179 pp. M. 4.40

LONEY (S. L.). Elements of hydrostatics. London, Clay, 1900. 12mo. 270 pp. (Pitt press mathematical series.). 4s. 6d.

ROSENBERG (F.). First stage mechanics of solids. For the elementary examination of the Science and art department. 3d edition. London, Clive, 1900. 12mo. 320 pp. (Organized science series.) 2s.

SLATE (F.). The principles of mechanics. An elementary exposition for students of physics. Part 1. New York, The Macmillan Co., 1900. 12mo. 10 + 299 pp. Cloth. $1.90

STREISSLER (J.). Elemente der darstellenden Geometrie für Oberrealschulen. 4te Auflage. Brünn, Winiker, 1900. 8vo. 4 + 162 pp., 4 plates. M. 2.60

VITI (R.). Sulla teoria matematica della previdenza; seconda conferenza tenuta il 17 marzo 1900 alla r. accademia dei ragionieri di Bologna. Roma, Giovanni, 1900. 8vo. 15 pp.

WILLIAMSON (B.) and TARLETON (F. A.). Elementary treatise on dynamics containing applications to thermodynamics, with numerous examples. 3d revised and enlarged edition. London and New York, Longmans, Green & Co., 1900. 12mo. 16 + 559 pp. Cloth. $3.50

ZEUNER (G.). Technische Thermodynamik. 2te Auflage; zugleich 4te Auflage der "Grundzüge der mechanischen Wärmetheorie." Vol. I: Fundamentalsätze der Thermodynamik; Lehre von den Gasen. Leipzig, Felix, 1900. 8vo. 16 + 436 pp. M. 13.00

LIBRARY.

MATHEMATICS AT THE INTERNATIONAL CONGRESS OF PHILOSOPHY, PARIS, 1900.

The first international congress of philosophy was held at Paris, August 1–5, 1900, under the presidency of M. E. BOUTROUX, professor at the Sorbonne. The proceedings of the third section, devoted to logic and history of the sciences, are of especial interest to mathematicians. The deliberations of this section were directed by Professor JULES TANNERY. In his inaugural address as president of the section, after the usual felicitations and customary acknowledgments, Professor Tannery emphasized the union between science and philosophy, a union which, he said, is better designated by the term unity. Any separation between scientists and philosophers is only apparent. The savants of the present century have set most important results before philosophers for speculation. Count the powerful geniuses who from the beginning of the century to its end have attacked the notion of space; measure how much the critique of this notion has done for the problem of knowledge. What illumination has been thrown upon the notion of function and, ascending farther, on the notion of number, by the revision of the principles of analysis! Will not the theory of ensembles show somewhat how we ought to think of infinity? Will not mathematical logic furnish us a sure instrument for the discernment and transformation of the data of intuition? Will not the algebra of logic enable us to penetrate deeper into the mechanism of deduction? Every hope seems legitimate, after what has been accomplished in the foundations of geometry. What surprises may be in store for us in the study of the principles of mechanics! Before turning aside from the mathematical sciences, Professor Tannery called attention to the import for philosophy of those works which had been devoted to the evolution and history of these sciences.

Among the papers which were presented to this section the following deal more or less directly with questions mathematical. They are given in the order of presentation. The résumés of the papers and the discussions occasioned by them are drawn from the number of the *Revue de Métaphysique et de Morale*, for September, 1900, which is specially devoted to the congress of philosophy.

G. VAILATI, "The classification of the sciences."

G MILHAUD, "On the origins of the infinitesimal calculus."

S. GÜNTHER, "On the history of the origins of the newtonian law of gravitation."

M. CANTOR, "On the origins of the infinitesimal calculus."

H. POINCARÉ, "The principles of mechanics."

B. RUSSELL, "The idea of order and absolute position in space and time."

H. MACCOLL, "Symbolic logic."

G. PEANO, "Mathematical definitions."

C. BURALI-FORTI, "The different logical methods for the definition of the real number."

A. PADOA, "Essay at an algebraic theory of integral numbers, with a logical introduction to any deductive theory whatever."

M. PIERI, "On geometry considered as a pure by logical system."

P. PORETSKY, "The theory of logical equalities with three terms."

E. SCHRÖDER, "An extension of the idea of order."

W. E. JOHNSON, "The theory of logical equations."

A. MACFARLANE, "The ideas and principles of the geometric calculus."

A. CALINON, "The rôle of number in geometry."

G. LECHALAS, "The comparability of various spaces."

J. HADAMARD, "On induction in mathematics."

R. BLONDLOT, "Exposition of the principles of mechanics."

M. LE VERRIER, "On the genesis and import of the principles of thermodynamics."

A. VASSILIEF, "Principles of the calculus of probabilities."

In his paper on the origins of the infinitesimal calculus, M. G. MILHAUD commenced by seeking these origins in antiquity; in the discovery of the incommensurable magnitudes, which destroyed the atomism of the pythagoreans; in the theory of ratios of Euclid, applied by Hippocrates to the quadrature of his lunes; finally, in the method of exhaustion of Eudoxus employed by Archimedes for the quadrature of the parabola. Passing to modern times, he discussed the method of indivisibles Cavalieri, as compared with the method of exhaustion of the ancients; then the problem of tangents and the methods proposed by Descartes and Roberval for solving it; finally, the problem of maxima and minima solved by Fermat. The paper concludes with an account of the contributions of Huygens and Barrow, and their respective relations with Leibniz and Newton.

M. Günther began his memoir on the origins of the newtonian law of gravitation by reviewing the early conceptions of weight, notably the nominalistic conception of Aristotle, which he opposed to the animistic conception and to the atomistic conception. It was against the peripatetic theory of heaviness and lightness that all the modern reformers struggled: Copernicus was not altogether free from it; it was Galileo who finally destroyed it. In the seventeenth century weight was conceived as a universal property of matter. Kepler conceived of gravitation as a magnetic attraction, and Roberval likened it to heat. On the other hand, Gassendi, Berigard, Digby, Hobbes, and Huygens attempted to explain it by corpuscular hypotheses. Up to this point only the law of masses was considered. The law of distances was formulated by Boulliau, but his explanation is bad; Borelli, on the contrary, correctly understood centripetal acceleration, but he did not find the formula. The author finds, in Leibniz's theory of verticity a precursor of Faraday's theory of lines of force. He comes finally to Hooke, the immediate predecessor of Newton, to whom the latter showed himself rather unfair. He concludes by declaring that the success of the newtonian law resulted in the triumph of the theory of action at a distance, in which Leibniz saw the restauration of the doctrine of occult qualities, and eclipsed for a time the corpuscular and kinetic theories, which are however returning to favor in our days.

M. Couturat added a historical note to M. Günther's memoir. The latter credited the Dutch physicist Deusing with having anticipated modern theories by distinguishing a potential effort from an actual force. M. Couturat remarked that this distinction between potential and actual is fundamental in the physics and metaphysics of Aristotle from whom Deusing and modern physicists have appropriated these terms.

The memoir of Professor M. Cantor on the origins of the infinitesimal calculus was read at a general session presided over by Professor H. Poincaré. The author recalled that the ancients had anticipated the integral calculus by their methods of quadrature and cubature. In the middle ages, Petrus de Daçia designated the continuous generation of magnitudes by the term "fluere"; Bradwardine distinguished two kinds of infinity; Nicole Oresme, from the consideration of latitudes and longitudes, discovered that the variation is zero at the points of maxima and minima.

Kepler rediscovered this theorem; he generalized the idea of Vieta that the circle is a polygon having an infinite number of sides infinitely small, and applied it to the quadrature of the circle and the cubature of the sphere and solids of revolution. Cavalieri conceived the method of indivisibles, which permitted him to effect veritable integrations. Descartes found a general method, purely algebraic, for calculating the normal to a curve, and consequently its tangent. But it is Fermat, the greatest French mathematician of the seventeenth century, who really anticipated the infinitesimal calculus by his method of maxima and minima; he also invented a method of tangents superior to that of Descartes. Roberval found another solution of the problem of tangents by the composition of movements; but if his method is ingenious, it is of little practical value, since at each application it demands a new exercise of ingenuity, and a good method ought to dispense with such demand once for all. After having spoken of Pascal's Traité des sinus du quart de cercle, of which a figure suggested to Leibniz his differential calculus, the author arrived at Barrow, the tutor of Newton, from whom the latter extorted the last of his Lectiones geometricae (1669). Leibniz invented his calculus October 29, 1675, a year before he saw the manuscript of Newton's Analysis per aequationes. While the priority belongs to Newton, the originality of Leibniz is incontestable. Moreover, the latter developed his method logically, while Newton was changeable in his conceptions, using limiting ratios to mask his infinitesimals. Finally, Newton did wrong in holding secret that which Leibniz threw to the winds.

In the discussion which followed the reading of M. Cantor's memoir, M. Milhaud remarked that M. Cantor had given the facility and generality of a method as the criterion of its value, and asked if Descartes, as the inventor of analytical geometry, ought not to be ranked above Fermat, whom Cantor had proclaimed to be the greatest French mathematician of the seventeenth century. M. Cantor replied that Descartes, in his Geometry, had not so much founded analytical geometry as the general theory of equations. Descartes is so little the inventor of analytical geometry that we do not find in his work the equation of the straight line, while Fermat was familiar with it. Analytical geometry is much older than either of these two savants; if we ask who wrote the first treatise on analytical geometry, it is De Witt. The superiority of Fermat's genius shines in the theory of numbers, which Dessumed by the principles of mechanics in M. Poincaré's exposition. They are conventions which experience can never bring to default: because as soon as any fact should contradict them we would always find, nolens volens, a means of adapting them to the new fact. For example, if in any case the principle of Kepler did not seem to be verified, we would explain the divergence by the existence of unsuspected facts, such as electrical and magnetic phenomena, etc., which would be manifested and measured by this divergence, and which would be the object of a new science. Without disputing the justness of M. Poincaré's conclusions, the speaker found them possessed of an excessive scepticism. The principles of mechanics are imposed by experience, they are the quintessence of innumerable experiences, crude or precise; and when they seem to become defective, the new facts that we are obliged to introduce to cover the deficiency assume of themselves a scientific character, that is to say, submit themselves to the principle of causation ; in a word they appear as true phenomena, and not as phantasms or fictions. On the contrary, to replace one of the principles by a different one is to be submerged by innumerable complications in the study of the most simple facts. For example, the law of gravitation is verified by a multitude of observations, but in other cases it appears at fault ; we explain this divergence by saying that the bodies in question are electrified, or magnetic, etc., and we measure these new phenomena precisely by the discrepancy between the true attraction or repulsion and the newtonian attraction. It would seem then that the law of Newton is only a convention that the facts never contradict, because when they seem to contradict it, we invent new facts to justify it. Still, who would dream of replacing Newton's law by the following convention : "Two bodies repel each other proportionally to their distance and inversely as their masses," correcting the divergence between this and experience by means of supplementary hypotheses? We feel that the law of Newton is a convention preferable to all others, because it is clearly imposed by the facts. Now the principles of mechanics are imposed by facts still more manifestly than the law of Newton. To sum up, M. Painlevé conceives physical science as a method of successive approximations, oriented initially by empiricism and guided by certain principles of experimental origin. The "convergence" of this method is not assured a priori, but well justified by its success, i. e., by the more and more natural and perfect accord between theory and reality. In seeking the laws of nature, it is the divergence and increasing complications which give warning that we have lost our way.

M. Poincaré replied that there was really no lack of accord between M. Painlevé and himself. He himself recognized that science has always proceeded and will always proceed by successive approximations. But he was anxious to point out the series of artifices, more or less conscious, by which the founders of mechanics had succeeded in transforming the first approximation, not into a provisional truth susceptible of correction, but into a definitive and rigorous truth; and this to a great improvement in clearness of statement, and consequently to the benefit of science itself.

M. Hadamard observed that if, with Kirchhoff, we assign as the object of mechanics, not the explanation of the phenomena of motion but merely their description in the simplest and most exact manner, the principles of this science, as we state them, are sufficiently justified. When we find facts in apparent contradiction with these principles we are perfectly justified in making a new force intervene, which is always found to account very simply for the phenomena, in place of changing the general principles and thus involving ourselves in contradictions with the aggregate of other known facts. Besides, according to a remark of M. Duhem, it is not a single determinate hypothesis, but the ensemble of the hypotheses of mechanics, that we can attempt to verify experimentally. As to the definition of force, M. Hadamard thought that we ought not to be satisfied with defining force as the product of mass and acceleration, because we do not thereby recognize one of the essential characteristics of force, namely, that it should represent the action of one body on another. To take account of this characteristic it is absolutely necessary to adjoin the principle of the independence of forces to that of inertia; the former is formulated thus: When a body is in the presence of several others, the acceleration which it experiences is the geometric sum of several segments of which each depends only on the state of the body influenced and that of one of the influencing bodies. The notion of "force exercised by one body on another," as introduced by this principle, is, moreover, necessary to the enunciation of the principle of the equality of action and reaction in all its generality; for by virtue of the definition of the internal forces of a system it can be applied to a non-isolated system.

In reply, M. Poincaré agreed that the experimental sciences can never verify anything but an ensemble of hypotheses. Every experiment furnishes us, so to speak, one equation in a very great number of unknowns. Our science, still imperfect, does not give us a sufficient number of equations; we have fewer than there are unknowns. We can count on new experiments to give us continually new equations, which will diminish the indeterminacy of the problem. But as regards the unknowns introduced by geometry (curvature of space) or by rational mechanics (its most general principles), there is something more. Not only does experience not give us enough equations to determine them, but it is absurd and contradictory to suppose that it can ever give them; for the reason that these unknowns enter into the experimental problems as auxiliary supererogatory variables. This explains why the hypotheses which one might make relative to these unknowns are neither true nor false. As to the principle of the independence of the effects of forces, M. Poincaré declared that it is not true. If a piece of iron be simultaneously subjected to the action of a magnet and another piece of iron, the effect experienced by it is not the geometric sum of those exerted upon it by the magnet and the second bit of iron separately. The principle can only be saved by a coup de pouce: we say that the second piece of iron is modified by the presence of the magnet.

M. Padoa said that the distinction between axioms and definitions has only a logical and subjective value; in the real world there are only facts all given on the same plane. There are no ideas more simple or more evident than other ideas; there are merely ideas not defined and propositions not demonstrated with respect to the logical system adopted. And this logical system can or cannot be verified by the facts, according to the interpretation given to the ideas not defined. In a word, M. Padoa maintains the mutual independence of the logical and the real.

M. Aars replied to M. Padoa to the effect that we encounter facts in the subjective and psychic world as well as in the real world. He held, contrary to M. Poincaré, that the axioms of mechanics ought to aim at the existence of mechanics, and consequently should be capable of being true or false, just as any other proposition relative to any existence whatever.

M. Poincaré replied that questions of existence of this nature seemed to him as devoid of meaning as those of the truth or objectivity of the principles of mechanics.

M. Ribert also protested against M. Poincaré's scepticism. He held that the laws of mechanics have an objective value, and that they are not creations of the human intellect. The world existed before humanity, and the world will exist after it. It already obeyed, and it will continue to obey, the laws of mechanics. Hence science is true in the sense that it deals with real existences.

M. Poincaré remarked that we raise here the question of the reality of the external world, which would be more in place in the first section (metaphysics).

Mr. Russell read extracts from his memoir on the idea of order and absolute position in space and time. After having distinguished the absolute series, that is to say those whose elements *are* positions, from relative series, whose elements *have* positions by correlation with those of an absolute series, the author defines the absolute theory of time according to which an event is dated by its relation to the instant when it exists, and the relative theory according to which it is dated only by relations of simultaneity or succession with other events. In order that the latter be tenable it is necessary that between two events, considered simply as qualities, there should exist a constant and determinate temporal relation, which is not the case. Similarly the simultaneity of several events can reduce to no property common to these events, unless it is the fact that they occupy the same instant. It is necessary then to admit temporal positions as absolute. Similarly it is necessary to admit spatial positions as absolute and for the same reasons. The author discusses the arguments invoked by Leibniz and Lotze in support of the relative theory of space. This theory contends that the position of a point is only the ensemble of its distances to other points, which supposes that the only relation which can exist between two points is their distance. But this is false; they have another relation which is the direction of the straight line (projective) which joins them. Without the latter there is no means of conceiving angle, which is a relation between two directions and not between two distances. The definition of a plane postulates a new fundamental relation (between three points). The author shows that the relative theory presents all the difficulties of the absolute theory, and reduces the theory of relations of Lotze to an absurdity. He finds the absolute theory to possess the advantages of logic, clearness and simplicity over the relative theory.

M. Tarde opened the discussion of the paper by contesting the analogy of space and time with respect to their relativity. It seems that space must be purely relative, while time has something of the absolute. In space the origin is arbitrary, in time it is not; the present instant differs qualitatively from the past, and still more from the future. Hence to say that two bodies occupy (successively) the same place has no meaning; while there is real meaning in saying that two events take place in the same time.

Mr. Russell replied that M. Tarde's distinction has only a psychologic interest but no logical value; to the consciousness time appears more real and more absolute than space; but for the theory of knowledge, time and space are altogether analogous.

M. Aars suggested that the question whether space and time are absolute or relative reduces to the question whether they are subjective or objective. Relation and relativity are purely subjective functions; if space and time are relative they will be subjective, if absolute they will be objective.

Mr. Russell declared in reply that he was concerned merely with a question of logic or epistemology that had nothing to do with the metaphysical question of the objectivity of space or the psychological question of the origin of knowledge, a question anterior and superior to each of these.

In a résumé of his memoir on symbolic logic and its applications Mr. MacColl stated that the essential element of pure logic is the proposition. Propositions arrange themselves into two classes: the true (τ) and the false (ι); or in fact into three classes: the variable (θ), which can be true or false; the certain (ϵ), which are always and necessarily true (probability 1), and the impossible (η), which are always and necessarily false (probability 0). The symbol A^x signifies that the proposition A appertains to the class x; the symbol A^{xy}, equivalent to $(A^x)^y$, signifies that the proposition A^x belongs to the class y. For example $A^{\theta\epsilon}$ states: "it is certain that A is variable." The product A^xB^y is the simultaneous affirmation of the two propositions A^x and B^y; the sum $A^x + B^y$ is their alternative affirmation, namely, that one or the other is true. In general, the exponent τ is suppressed and the exponent ι is replaced by an accent (sign of negation). We have the formulæ:

$$(AB)' = A' + B', \quad (A + B)' = A'B', \quad A(B + C) = AB + AC,$$

$$(AA')^{\eta}, \quad (A + A')^{\epsilon}, \quad (A^{\epsilon} + A^{\theta} + A^{\eta})^{\epsilon},$$

$$A^{\theta} = A^{\epsilon\iota}A^{\eta\iota}, \quad A^{\iota\epsilon} = A^{\eta}, \quad A^{\iota\eta} = A^{\epsilon}.$$

The notation $(A : B)$ affirms that the proposition A implies the proposition B. It is equivalent to each of the assertions

$$(AB')^{\eta}, \quad (A' + B)^{\epsilon}.$$

The equality $(A = B)$ is equivalent to the two inverse implications $(A : B)$, $(B : A)$. For simplicity, the implication $(A : B)$ is often written A_B.

If we multiply the two certitudes

$$(A + A')(A^{\epsilon} + A^{\theta} + A^{\eta}),$$

we find the following alternative (certain)

$$A^{\epsilon} + AA^{\theta} + A'A^{\theta} + A^{\eta} = A^{\epsilon} + AA^{\epsilon'} + A'A^{\eta'} + A^{\eta},$$

whose four terms signify respectively: 1° that A is always true; 2° that A is sometimes true, but not always; 3° that A is sometimes false, but not always; 4° that A is always false. The author thus comes upon the four classic forms of propositions (A, I, O, E).

The author defines and calculates on one hand the weakest premise from which we can deduce a given proposition; and on the other hand the strongest conclusion that can be drawn from a given proposition.

He applies his symbolic logic to the calculus of probabilities. The symbol $\frac{A}{B}$ represents the probability that A be true when B is true, that is to say the relative probability of A with respect to B. The symbol $\frac{A}{\epsilon}$ represents the probability of A with respect to the data of the problem (regarded as certain), that is the absolute probability of A.

If $\frac{A}{B} = \frac{A}{\epsilon}$, we say that the probability of A is independent of B, since it does not change when we add B to the data of the problem. Consequently we take as measure of the dependence of A relative to B the difference

$$\delta\frac{A}{B} = \frac{A}{B} - \frac{A}{\epsilon}.$$

It is proved that if A is independent of B, B is also independent of A.

Compound probabilities are evaluated as functions of simple probabilities by means of the following formulæ:

$$\frac{AB}{\varepsilon} = \frac{A}{\varepsilon} \cdot \frac{B}{A} = \frac{B}{\varepsilon} \cdot \frac{A}{B},$$

$$\frac{A+B}{\varepsilon} = \frac{A}{\varepsilon} + \frac{B}{\varepsilon} - \frac{AB}{\varepsilon},$$

$$\frac{A'}{\varepsilon} = 1 - \frac{A}{\varepsilon}, \quad \delta\frac{A'}{B} = -\delta\frac{A}{B}.$$

If we put

$$\frac{A}{\varepsilon} = a, \quad \frac{B}{\varepsilon} = b, \text{ etc.},$$

we have

$$\frac{A}{B} = \frac{a}{b} \cdot \frac{B}{A}, \quad \frac{A}{B'} = \frac{a}{b'} - \frac{b}{b'} \cdot \frac{A}{B},$$

$$\delta\frac{A}{B} = \frac{a}{b}\delta\frac{B}{A}, \quad \delta\frac{A}{B'} = -\frac{b}{b'}\delta\frac{A}{B}, \quad \delta\frac{A'}{B'} = -\delta\frac{A}{B'} = \frac{b}{b'}\delta\frac{A}{B}.$$

By means of these formulæ it is easy to pass from a problem in logic to one in probabilities.

In the discussion occasioned by Mr. MacColl's paper, M. Couturat remarked that the calculus of probabilities had inspired both the system of MacColl and that of Boole; but the former, by his definition of the independence of two propositions, had been able to correct an error committed by Boole in a problem of probabilities. But the origin of Mr. MacColl's logic is found in a purely mathematical problem, namely, the calculation of the limits of a multiple integral when the order of the integrations is changed, a problem to which the last part of Mr. MacColl's memoir is devoted.

M. Peano read a memoir on mathematical definitions, in which he said that a definition is an equality whose first member contains the expression to be defined and the second member is composed of well-known terms. It supposes, then, a certain number of known terms which ought to be tabulated; the value of a definition is essentially relative to this table. M. Peano analyzed, by way of example, the first definitions of Legendre's geometry and some of Euclid's definitions. He shows that they define relatively simple and clear terms (line, point, straight line, surface) by means of terms which are less so (length, breadth, thickness, extremity, path, shortest), or a term (number) by a synonymous term (ensemble). He calls equalities of the form specified possible definitions. An idea will be definable or undefinable according to the order assigned to the ideas. It is convenient to arrange the ideas of a science in an order such that the number of primitive ideas relative to this order shall be as small as possible.

A definition should be complete, that is, intelligible by itself; and homogeneous, that is, the two members should contain the same variable terms. Thus, the formula

$$0 = a - a$$

is not a definition because we do not know what a is. The complete proposition: "Let a be any number; then $0 = a - a$," is not a definition because it is not homogeneous (the variable a enters into the second member but not into the first). It is necessary to write: "0 = the constant value of the expression $(a - a)$, whatever be the number a." In this proposition the letter a is an apparent variable, since the second member does not depend on its value.

M. Peano criticised other non-homogeneous definitions by showing that we can draw false conclusions from them (for example, the definition of operations on fractions).

M. J. Tannery remarked that if every definition is an equality, it is necessary to place the idea of equality in the table of undefined ideas.

M. Schröder thought that it is unnecessary to impose too restrictive conditions on definitions. For example, the zero (nothing) of logic can be defined by the formula

$$0 = a\,a_1$$

whatever be a, where a_1 is the negation of a, or not-a.

M. J. Tannery took exception to the definition of the zero of arithmetic by the formula

$$0 = a - a,$$

whatever be a. In this formula there enters the sign of subtraction. Now we can define subtraction only in the case of unequal numbers. In other words, we define 0 as the number which when added to a gives a as sum. But, by hypothesis, we know no such number: every number added to another number gives a sum different from each of them. Then this does not give the idea of zero, which seems to be a primitive and indefinable idea.

M. Peano replied that the formula

$$0 = a - a$$

does not constitute a definition, since it is not complete, it being necessary to add what a is. Nor is it homogeneous, the second member containing a term which does not enter the first. It is necessary then to say: "$0 =$ the constant value of $(a - a)$ whatever be a." The same observations apply, mutatis mutandis, to the analogous definition of the zero of logic. As to the equality definition, it is undoubtedly among the primitive ideas; but it differs from special equalities, which can consequently be defined.

M. Padoa emphasized a distinction between the formula "$0 = a - a$" and the formula "$0 =$ the constant value of $(a - a)$ whatever be a." The first supposes a mathematical fact known, namely, that $(a - a)$ is constant independently of a. The second, on the contrary, implies this fact and expresses no more than a purely logical fact: the equality of two constant values.

M. Couturat analyzed the memoir of M. BURALI-FORTI upon the different logical methods for the definition of a real number. The author distinguishes three kinds of definitions: 1° the nominal definition, which consists of equating the new sign (to be defined) to an expression composed of known signs; 2° the definition by postulates, which consists in enumerating the fundamental relations verified by a group of primitive ideas; 3° the definition by abstraction, which consists in defining a function f by saying in what cases we have $fx = fy$.

These distinctions made, all the definitions given of an integral number enter into one of the three classes. The nominal definition of an integral number rests upon that of homogeneous magnitude. "If $+$ is a binary similar operation, we say that u is a *class* homogeneous with respect to $+$ if, x and y being any elements of u, $x + y$ is a determinate element of u." "Let u be a homogeneous class with respect to $+$. We call zero the element x of u such that, whatever element of uy may be, we have always

$$y + x = y."$$

We define then the ensemble N_0 of positive integers and zero as a class of operations. Let x be any magnitude of the class u. We define the class of magnitudes N_0x (multiples of x) by the following properties:

1° N_0x is homogeneous with respect to $+$;

2° 0 and x are elements of N_0x;

3° Every N_0x which is not equal to zero is of the form $y + x$, y being an N_0x. We can then define nominally the zero of arithmetic and the notion of successive integers, whence are deduced the five characteristic properties of integral numbers.

The definition by postulates (Peano-Dedekind) consists in putting these five properties as postulates. The definition by abstraction (G. Cantor) consists in defining the cardinal number by the conditions of equality between two cardinal numbers.

The memoir terminates with nominal definitions of rational numbers and real numbers.

M. PADOA presented the principal ideas of his introduction to any deductive theory. Every deductive theory rests upon a system of undefined symbols and a system of improved principles. The logical value of the deduction is independent of the meaning of these symbols; the system of ideas that we make correspond to them is only an interpretation. The logical value of a theory, then, is absolutely independent of empirical and psychological data; and, though the concrete interpretation is useful, it is sometimes dangerous in that it covers up gaps in the reasoning. The undefined symbols may have several interpretations which verify the undemonstrated principles; these interpretations are logically equivalent (e. g., the principle of duality in projective geometry).

The author defines, and gives the ordinary criterion for, an irreducible system of undemonstrated principles. He shows, moreover (and this is the personal contribution of the author to this theory), that the system of undefined symbols is irreducible with respect to the system of undemonstrated propositions when it is not possible to deduce from the latter the symbolic definition of any of the undefined symbols. To show this irreducibility it is sufficient to find an interpretation of the system of undefined symbols, such that they continue to verify the system of undemonstrated propositions when we change the sense of any one of them, and this for every one of the undefined symbols taken separately. These are the principles which M. Padoa applies to the theory of integral numbers (positive, negative and zero) founded upon three undefined symbols *integer*, *successive to*, *symmetric of*, and upon seven undemonstrated propositions, including the principle of complete induction understood in both senses.

The memoir of M. PIERI on geometry considered as a purely logical system was read in abstract by M. Couturat.

Geometry, stripped of every intuitive element, is an ideal and deductive science, which has as its object a certain order of logical relations. Like arithmetic it is becoming more and more abstract; the idea of point or spatial element is generalized, as well as that of space. Notwithstanding its empirical origin and the pedagogical and practical value of its applications, geometry is being little by little shorn of all geometric meaning in order to become a hypothetico-deductive system. Such a system ought to rest upon an aggregate of primitive propositions and primitive concepts (the latter have been much less studied than the former). M. Pasch has reduced the primitive concepts of geometry to four: point, plane, segment, and congruence. M. Peano has reduced them to three: point, segment, and movement. M. Pieri proposes to reduce them to two only: point and movement. He also attempts to restrict the rôle of movement, which he reduces to a relation between four points (congruence of the two couples AB, CD) and further to a relation between three (congruence of the two couples AB, AC).

The choice of primitive ideas (which cannot be guided only by vague considerations of simplicity or of evidence) ought to be regulated by the following criterion: every science being characterized by a maximum group of transformations which do not alter the properties that it studies, the choice ought to be so made that the primitive ideas should be invariants with respect to this group. As examples of trespassers against this rule of logic the author cites the idea of length in analysis situs, that of a half-ray in projective geometry, and even the general ideas of line, surface, solid, and space in elementary geometry. The memoir terminates with the enumeration of twenty postulates which serve as a logical basis for geometry and which define successively, by means of the idea of movement, the straight line, plane, perpendicularity, and finally segment.

As an introduction to M. Poretsky's memoir on the theory of logical equalities having three terms M. Couturat gave an outline of the principles of Schröder's logical calculus.

The fundamental relation of the algebra of logic is the relation of inclusion. We reduce the concepts to their extension, that is, to classes or aggregates of corresponding objects; a is said to be in b and we write

$$a < b$$

if the ensemble a is contained in the ensemble b. It is the translation of the universal affirmative proposition "all a is b."

Similarly, propositions are reduced to their extension or to their domain of validity, that is, to the ensemble of instances or cases where each of them is true. If a and b represent propositions, $(a < b)$ signifies "every time that a is true. b is true," or "if a is true, b is true," or finally "a implies b."

Thus every formula of the algebra of logic is susceptible of two interpretations, one in concepts and the other in propositions. At bottom this algebra is simply the calculus of ensembles, what Leibniz would call the theory of the containing and the contained. To this fundamental relation of inclusion we join that of equality, which is defined formally thus

$$(a = b) = (a < b)\ (b < a),$$

that is, every equality is equivalent to two inverse inclusions having the same members.

Two undemonstrated principles are admitted: 1° the principle of identity

$$a < a\,;$$

2° the principle of the syllogism

$$(a < b)(b < c) < (a < c).$$

Logical addition and multiplication are formally defined by the following expressions:

$$(a < c)(b < c) = (a + b < c),$$
$$(c < a)(c < b) = (c < ab),$$

whence we conclude that $(a + b)$ is the ensemble of elements which contains the two classes a and b united, and that ab is the ensemble of elements common to two classes a and b. We define two special classes, the class 1, which contains all the others; the class 0, which is contained in all the others, and consequently contains itself no element. Finally negation is defined by the formulae

$$aa_1 = 0, \quad a + a_1 = 1,$$

of which the first translates the principle of contradiction, and the second the principle of excluded middle; which proves, in passing, that these two principles are independent of the principle of identity $(a < a)$.

The algebra of logic tends to transform the inclusions (which transcribe the verbal propositions directly) into equalities (more manageable for the calculus). To this end serve the two formulæ (known to Leibniz)

$$(a < b) = (a = ab),$$
$$(a < b) = (ab_1 = 0).$$

The second permits also of transforming any equality into an equality whose second member is zero

$$(a = b) = (ab_1 + a_1b = 0).$$

Every system of premises can be reduced to a single equality with second member zero, which will be the equation of the problem. This equation can be solved with respect to one of the variables, it matters not which. With regard to the unknown x it has the form

$$(ax + bx_1 = 0) = (ax = 0)(bx_1 = 0).$$

But

$$(ax = 0) = (x < a_1),$$
$$(bx_1 = 0) = (b < x).$$

Therefore

$$(ax + bx_1 = 0) = (b < x < a_1).$$

The equation is equivalent then to a double inclusion : x contains b and is contained in a_1. Such is the solution of the problem with respect to x.

In his memoir on the theory of logical equalities having three terms, M. Poretsky employs an original method of which the following are the principles.

The double inclusion

$$b < x < a$$

he expresses by the equality

$$x = ax + bx_1,$$

which is in fact equivalent to the two equalities

$$(x = ax) = (x < a),$$
$$(bx_1 = 0) = (b < x).$$

The system of M Poretsky rests upon another equivalence

$$(a = ab_1 + a_1b) = (b = 0),$$

paradoxical enough because the first member contains a letter a which the second does not. Hence a is absolutely indeterminate in this formula. Any equality

$$A = B$$

is equivalent to each of the following equalities :

$$(AB_1 + A_1B = 0) = (N = 0),$$
$$(AB + A_1B_1 = 1) = (N_1 = 1).$$

M. Poretsky calls N the complete logical zero and N_1 the complete logical whole of the problem represented by the equality $(A = B)$. Consequently being given any class U, the equality in question becomes

$$(A = B) = (U = UN_1 + U_1N) = (N = 0).$$

This is the law of forms of logical equalities, that is to say, the rule which shows how to find all the equalities equivalent to a given equality.

M. Poretsky employs for this purpose an exhaustive method, which consists in forming all the possible classes of discourse of n letters. Being given n simple classes a, b, c, d, $\cdots$, we can form with them 2^n elements (constituents of Boole) which the author calls the minima of discourse. To obtain all possible classes it is sufficient to form the additive combinations of the 2^n elements of which the number is 2^{2^n}, including 0 and 1. If we take for U each of these classes, we obtain 2^{2^n} different forms for each equality.

Any equality whatever consists in equating to zero the sum of m elements, that is, in annulling them separately. It is equivalent then to m elementary equalities, and to all their combinations 2^m in number, including the equality in the m elements themselves and the identity $0 = 0$. This is the law of consequences.

On the other hand, any equality in m elements can be considered as the consequence of an equality containing at least these m elements (with others). Hence all the possible causes of the given equality are obtained by joining to it all the combinations of the $2^n - m$ other elements, of which the number is $2^{2^n - m}$. This is the number of different causes of the equality, including the equality itself and the absurdity $1 = 0$. This is the law of causes.

M. Poretsky sums up the law of forms, the law of consequences, and the law of causes in a single table which includes the 2^{2^n} classes of discourse, from which may be written, by simple inspection without calculation, all the forms, consequences, and causes of the equality to which the table refers.

M. Couturat remarked in concluding his abstract that M. Peano was the first to determine the number of propositions that can be affirmed relative to n simple classes. As we have seen, the number of independent propositions is $2^{2^n}-1$, excluding the absurdity $1 = 0$. But we can affirm the alternative of any number of these propositions; there are as many alternatives as there are additive combinations possible among these propositions, that is,

$$2^{2^{2^n}-1}-1,$$

excluding again the absurdity $0 = 1$.

For $n = 2$, we find the number 32,767, or 32,766 if with M. Peano we exclude the insignificant identity $1 = 1$. Thus, with two terms a and b and their negations, we can construct 32,766 different judgments. We thus see the richness and complexity of the combinations permitted by the algebra of logic.

M. Schröder gave a rapid summary of the principles of the algebra of relations before presenting his memoir on an extension of the idea of number. The algebra of relations is the logic of relative terms, while the classic logic treats only of absolute terms. The proposed generalization of order he calls gradation. It consists in admitting several elements of the same rank. A gradation will be expressed by two relations; the one ρ responding to the double question "What element precedes what element??" the other, γ, responding to the double question "What element is of the same rank as what element??"

The essential characteristics of a gradation are the following:

1° Given any two elements i, j, either i is inferior to j, or i is superior to j, or i is of the same rank as j;

2° If i is inferior to j, it is not of the same rank;

3° On the same hypothesis, j is not inferior to i;

4° If i is of the same rank as h and h inferior to j, then i is inferior to j;

5° If i is inferior to h and h inferior to j, then i is inferior to j.

These five properties of any gradation can be expressed by the following formulæ:

$$1 = \gamma + \rho + \breve{\rho}, \quad \rho(\gamma + \breve{\rho}) = 0, \quad (\gamma + \rho) \;;\; \rho < \rho,$$

where $\breve{}$ signifies the converse of a relation, and $x \;;\; x < x$ signifies that the precedent of a precedent is a precedent; the first formula transcribes the property 1°; the second the properties 2° and 3°

$$\rho\gamma = 0, \qquad \rho\breve{\rho} = 0\,;$$

and the third the properties 4° and 5°

$$\gamma \;;\; \rho < \rho, \qquad \rho \;;\; \rho < \rho.$$

From these formulæ we can deduce all the properties of two relations γ and ρ: they are transitive, γ is symmetric ($\gamma = \breve{\gamma}$), and copulative ($\gamma \;;\; \gamma = \gamma$). Every gradation furnishes thus a root of the equation

$$x \;;\; x = x.$$

To exhibit the logical importance of his researches M. Schröder remarked that we do not know whether we can affirm that every ensemble is capable of being arranged in a simple order, and that the solution of this fundamental question of order depends upon the algebra of relations.

Mr. Macfarlane remarked relative to M. Schröder's memoir the difference existing between the algebra of logic, where the relations $=$ and $<$ are transitive, and the algebra of relations in which transitive relations constitute only a particular case. The algebra of relations serves as a connecting link between the symbolic logic and various branches of mathematics such as the calculus of operations and the geometrical calculus.

M. Couturat added that the theory of substitutions is a special branch of the algebra of relations. The product of two substitutions is their relative product; the inverse (s^{-1}) of a substitution is the converse ($\breve{s}$); the identical substitution (1) is the identical relation 1.

The memoir of Mr. Johnson on the theory of logical equations was presented in abstract by Mr. Russell. The author proposes to solve symmetrically the general logical equation in n unknowns by means of n indeterminates. To this end he solves not with respect to the unknowns (x, y, z) themselves, but with respect to their constituents xyz, $xy\bar{z}$, $\bar{x}yz$, $\cdots$. But the latter can be replaced by any divisors possessing the same properties, namely those of being mutually exclusive and collectively exhaustive, that is, verifying the conditions

$$\sum\rho = 1, \quad \sum\rho\rho' = 0.$$

The equations of logic are solved with respect to these divisors ρ. In passing from this solution to the solution relative to the unknowns x, y, z, $\cdots$, a dissymmetry is introduced which the author has attempted to remove. He expresses in conclusion his acknowledgments to the works of Schröder and Whitehead on the algebra of logic.

The paper of Mr. MACFARLANE on the ideas and principles of the geometric calculus is in abstract as follows: By geometric calculus he means that branch of algebra which is founded upon the properties of space of three dimensions. The researches of Hamilton ending in the invention of quaternions have their origin in Kant's theory of knowledge; proceeding from the idea that algebra is the science of pure time, he was finally led to the true spatial generalization of algebra. The algebra of quaternions bears the same logical relation to the algebra of complex quantities as does the latter to ordinary algebra. The fundamental rules of quaternions and those of vector analysis can be harmonized by modifying those of the latter so as to render them associative; the rules thus obtained form the special rules of reduction of the geometric calculus. The geometric calculus has been extended in such a manner as to include the trigonometry of surfaces of the second order. The fundamental laws of algebra have no need of modification to be applicable to plane algebra, that is to say, to the algebra of complex quantities; but modification is necessary in order to be applicable to the algebra of space, since the axis of the plane is then variable. In the geometric calculus the principal theorems, such as the binomial theorem, exponential theorem, and Taylor's theorem, remain true, provided the relative order of the vector symbols be preserved. The philosophical conclusion of the memoir is that the process by which the science of algebra has been extended is not an arbitrary convention, but a patient logical generalization tending to a determinate end and appropriate to the object to be represented.

M. CALINON read an account of his memoir on the rôle of number in geometry. Whatever may have been historically the geometric origin of this or that extension of number, the generalization of number ought and can be made independently of any intuitive consideration. As soon as the numerical continuum is constituted, namely the ensemble of real numbers, it can serve to define the geometric continuum. The line (continuous) is defined by establishing a one to one correspondence between the ensemble of real numbers and a series of points; the surface by setting up such a correspondence between the ensemble of real numbers and a series of lines; the solid, by establishing an analogous correspondence between the ensemble of real numbers and a series of surfaces. Thus it is the numerical continuum, simpler and clearer, which generates the geometric continuum, and number is essential to the definition of all geometric magnitudes. Projective geometry is independent only in appearance, for it implies a continuity which reduces in the final analysis to the numerical continuum.

The following is a brief résumé of the memoir of M. LECHALAS on the comparability of various spaces. There are those who deny to general geometry the right to consider of one or several spaces as contained in a space of four dimensions, and to conceive of several coexistent spaces. Nor do they allow the identification of euclidean spheres and Riemannian planes or euclidean pseudo-spheres and Lobachevskian planes. M. Lechalas first set aside the purely verbal objections founded on the use of the word space to designate varieties included the one in the other. If, in a plane four-dimensional space, we consider a spherical space of three dimensions and a euclidean plane space of three dimensions passing through the center of the former, their intersection is a surface (two-dimensional variety) which possesses, in the euclidean space, all the properties of the euclidean sphere and in the spherical space, all the properties of the Riemannian plane. Why refuse to admit the identity of this euclidian sphere and this Riemannian plane, since they are the same figure? It is maintained that it possesses different properties in the two spaces. Unquestionably, as the circle (intersection of a plane and sphere) possesses different properties on the two surfaces: in the plane it has but one center; on the sphere it has two; if the plane passes through the center of the sphere the circle will be a geodesic on the sphere without being a geodesic of the plane. But this does not prevent it from being the same figure in both cases. Besides the partisans of the incomparability of spaces are logically driven to maintain the indiscernibility of Riemannian spaces among themselves and of Lobachevskian spaces among themselves, and consequently to admit only three types of spaces qualitatively heterogeneous and foreign to one another (euclidean, Riemannian, Lobachevskian). They thus suppress the continuity which binds these three spaces together, and which makes the euclidean geometry a limiting case of the other two; they thus destroy the unity of general geometry and its philosophical character.

In discussing the paper of M. Lechalas, M. Peano remarked that we can establish for four-dimensional space propositions which are not valid for space of three dimensions. In fact they rest upon an additional postulate. To the three following postulates

1°. If a is a point, there are other points than a;

2°. If l is a straight line, there are points outside of l;

3°. If π is a plane, there are points not in π;

we ought to add the postulate

4°. If Σ is a space of three dimensions, there are points not in Σ.

It is easily seen that there are propositions which result from these four postulates taken together and which do not result from the first three. Conversely such a theorem as the property of homological triangles is true for the plane and three dimensional space but does not hold for space of four dimensions, because it depends on the number of dimensions of the space.

M. Hadamard expressed the opinion, in the discussion, that space of four dimensions is only an analytical ensemble; and that general geometry is only a geometrical interpretation, though a legitimate one, of analytical facts and formulæ.

Mr. Russell said that a plane space of two dimensions is not identical with a plane of a space of three dimensions; the latter enjoys special properties which it possesses because situated in a space of higher dimensions. One of the proofs is that projective geometry even of the plane demands a space of three dimensions for its construction; in fact, to prove that the construction of von Staudt's quadrilateral is a univocal operation (with a unique result) it is necessary to be able to effect it in two different planes. Besides, in order that the essential duality of projective geometry shall be complete, a space of at least three dimensions is necessary. There is then a specific difference (of nature, and not of degree) between spaces of two dimensions and those of three dimensions.

Mr. Russell objected that M. Lechalas passes from the identity of the analytical properties of two spaces to the conclusion of their real identity. This is to ignore the difference between analysis and geometry. From an analytical point of view we conceive whatever we wish: one only constructs algebra and has need to speak neither of points nor of figures. But to construct a geometry it is necessary to get a space to which we can apply the analytical formulæ, and in this space planes and straight lines which correspond actually to their projective definition (namely, of being determined by three or two points). We cannot apply the abstract projective geometry to any figure, for example, take for plane any surface whatever; to take a sphere is to suppose the whole of the metrical geometry. To be sure, the geodesics on a sphere are determined by two points, but this presupposes that the sphere itself has been determined by its distance from an exterior center or by some other equivalent metrical property. The sphere then can never be likened to a plane, not even to a non-euclidean one.

M. HADAMARD read a résumé of his memoir on induction in mathematics. The method of invention in mathematics is generalization. What is the value of this process? what is the probability of its success? The author thinks that it often fails; not only is there no reason why generalization should succeed, but there are often reasons for it not to succeed. If a certain problem has been solved in a certain particular case, this is generally due to the fact that this particular case offers some special property that renders it simpler and easier to treat. We cannot extend this property to the general case to resolve it in an analogous manner. The recent progress of mathematics offers plenty of examples in support of this statement. The more science advances, the more the part of chance in the discovery of truth diminishes; the more, consequently, should induction and analogy be found misleading. Being given two analogous problems, one of which has been solved and the other not, there is occasion to think that the results found in the solution of the first are very different from those to be obtained in the solution of the second.

The memoir of M. BLONDLOT on the principles of mechanics was read by M. Couturat. The author has followed Kirchhoff and Mach, and the object of his work is didactic rather than critical. He marks sharply the distinction between ideal and fictitious mechanics and real and positive mechanics. The former supposes an ideal system of data (refères) and an absolute clock. Admitting the notion of material point, the principles of mechanics may be formulated as follows:

1°. A material point, supposed alone, can assume no acceleration.

2°. Two material points determine on each other accelerations directed in opposite senses along the line joining them.

3°. The ratio of the numerical values of the accelerations produced by two given material points A and B is constant.

The latter principle permits of defining and measuring the mass of a material point by means of the following addition: "X being any material point, the ratio of the mutual accelerations of A and B is equal to the ratio of the mutual accelerations of A and X divided by the ratio of the mutual accelerations of B and X. Whence the ratio of the masses of two material particles will be, by definition, the inverse of the ratio of their mutual accelerations."

4°. The acceleration induced in any material point M by an aggregate of several material particles S, S', S'', $\cdots$ is obtained by compounding, by the rule for the composition of vectors, the accelerations determined separately by S, S', S'', $\cdots$, when acting successively on M. The notion of force is useless, but, to abbreviate, we can define it as the product of the mass by the acceleration.

Theoretical mechanics is to positive mechanics as a model to an actual construction. To apply the former to the latter we choose a system of data and a time-piece in such a manner as to obtain the most exact description possible of actual movements. This is undoubtedly inexact; but if it is approximately exact, it is simply because theoretical mechanics, fictitious and conventional as it is, was invented to be applied to reality.

In the discussion M. Kozlowski was of the opinion that the idea of force should not be banished from mechanics. Du Bois-Reymond has shown that impact is inconceivable, and that action at contact can be explained only by action at a distance. If science substitutes acceleration for force it is because it tends to replace muscular symbols by visual symbols.

M. Bulliot would not agree to replace force by acceleration under the pretext that force is an obscure idea taken from our common experience. Why substitute for notions a little obscure but rich in content, others clear but barren? Modern science through an excessive love of logical clearness tends to lose all contact with reality to which it owes its origin.

M. J. Tannery replied that if science replaces experimental notions and the confused data of sense by abstractions, it is to satisfy a need of rigor and precision ; but it does not thereby lose all contact with reality, and applied science is benefited by the progress of pure science.

M. Foveau de Courmelles did not admit the distinction between statical force and accelerating force : both correspond to the same psycho-physiological impression of effort.

M. Vassilief confirmed the ideas of M. Kozlowski upon the difference between visual and muscular sensations. The privileged rôle of visual sensations in science is due to the fact that they are the only ones which are exactly measurable. The speaker recalled Lobachevsky's definition of time : time is a motion suitable to measure other motions.

In M. VASSILIEF's memoir on the principles of the theory of probabilities he distinguishes three kinds of probabilities : 1° subjective probability, simple sentiment of expectation or hope ; 2° mathematical probability, defined numerically as the ratio of the number of favorable cases to the number of possible cases ; 3° objective or physical probability, which results from the observation of a very great number of experimental cases. We know that the last approaches mathematical probability as the number of observed cases increases. M. Vassilief asks whether there is not an analogous relation between subjective and mathematical probability. They do not seem to him to be proportional ; he believes rather that they obey Fechner's law. In concluding he renders homage to the French philosopher Cournot, who, in his opinion, has expressed most profound views on the theory of probabilities.

In the discussion M. Couturat took exception to the value and sense of Fechner's law, and was of the opinion that subjective probability has no other measure than mathematical probability, and that, consequently, it is out of place to raise a question of proportionality between them.

M. Dickstein cited the works of Goziewski (analyzed in the *Jahrbuch uber die Fortschritte der Mathematik*), and stated that M. Petzoldt (*Vierteljahrschrift für wissenschaftliche Philosophie*) had attempted to apply the theory of probabilities to the principles of mechanics.

E. O. LOVETT.

PRINCETON UNIVERSITY.

A DEMONSTRATION OF THE IMPOSSIBILITY OF A TRIPLY ASYMPTOTIC SYSTEM OF SURFACES.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, December 28, 1900.)

Consider in space any system whatever of curvilinear coördinates, ρ_1, ρ_2, ρ_3 and let the Cartesian coördinates x, y, z of a point with respect to fixed rectangular axes be given in terms of the preceding by the equations

$$(1)\quad x = f(\rho_1, \rho_2, \rho_3), \quad y = \varphi(\rho_1, \rho_2, \rho_3), \quad z = \psi(\rho_1, \rho_2, \rho_3).$$

It is evident that the coefficients of the system of equations

$$(2)\quad \begin{aligned} \frac{\partial^2\theta}{\partial\rho_1^2} &= a_{11}\frac{\partial\theta}{\partial\rho_1} + a_{12}\frac{\partial\theta}{\partial\rho_2} + a_{13}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_2^2} &= a_{21}\frac{\partial\theta}{\partial\rho_1} + a_{22}\frac{\partial\theta}{\partial\rho_2} + a_{23}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_3^2} &= a_{31}\frac{\partial\theta}{\partial\rho_1} + a_{32}\frac{\partial\theta}{\partial\rho_2} + a_{33}\frac{\partial\theta}{\partial\rho_3} \end{aligned}$$

can be so determined that it will admit x, y, z, as particular simultaneous solutions.

Any point (x, y, z) may be looked upon as the intersection of three surfaces, one of each of the families for which ρ_1, ρ_2, ρ_3 are the respective parameters.

Consider a surface of each family and the point defined by them. From the form of equations (2)* it is evident that the necessary and sufficient condition that the surfaces $\rho_1 =$ const. and $\rho_2 =$ const. cut out upon the surface $\rho_3 =$ const. the asymptotic lines through the given point, is expressed by

$$a_{13} = a_{23} = 0.$$

Since similar results are obtained for the other two surfaces through the point, and in turn for every point and the surfaces determining it, it follows that the necessary and sufficient condition that the surfaces of a triple system cut one another along asymptotic lines is that the coefficients of the equations (1) satisfy the condition

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* Bianchi, Lezioni, . 109.

$$(3) \qquad a_{13} = a_{23} = a_{12} = a_{32} = a_{21} = a_{31} = 0.$$

The square of the linear element on the surface $\rho_3 =$ const. through a given point may be written in the form

$$(4) \qquad ds_3^2 = E_3 d\rho_1^2 + 2F_3 d\rho_1 d\rho_2 + G_3 d\rho_2^2,$$

where

$$E_3 = \Sigma\left(\frac{\partial x}{\partial \rho_1}\right)^2, \quad \Sigma F_3 = \frac{\partial x}{\partial \rho_1}\frac{\partial x}{\partial \rho_2}, \quad G_3 = \Sigma\left(\frac{\partial x}{\partial \rho_2}\right)^2,$$

and where, by hypothesis, the parametric lines are the asymptotic lines.

Denoting by $\begin{Bmatrix} rs \\ t \end{Bmatrix}_3$ the Christoffel symbols formed with respect to (4) we have

$$a_{12} = \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_3, \quad a_{21} = \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_3,$$

Hence, by (3),

$$\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_3 = \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_3 = 0,$$

that is,* the ρ_1 and ρ_2 lines are geodesics; but this is impossible, since they are asymptotic lines. Since similar results follow for surfaces $\rho_1 =$ const. and $\rho_2 =$ const., we have the theorem

There is no triple system of surfaces cutting mutually in the asymptotic lines of these surfaces.

This theorem leads to several interesting negations.

Consider a triple system of surfaces cutting one another in a system of conjugate lines on each. Suppose now that we have another triply conjugate system of surfaces, which are such that to each point determined by three surfaces of the first system there corresponds a point determined by three surfaces of the second system and in such a way that the respective surfaces defining the first point correspond by orthogonality of linear elements to those defining the second point. From the theory of infinitesimal deformation of surfaces we know that if (x, y, z) and (x_1, y_1, z_1) denote corresponding points in these two systems, then the triple system of associate † surfaces of the first system are defined by the equations ‡

$$d\xi = z_1 dy - y_1 dz, \quad d\eta = x_1 dz - z_1 dx, \quad d\zeta = y_1 dx - x_1 dy,$$

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* Bianchi, Lezione, p. 146.

† Ibid., p 279.

‡ Genty, *Ann. de la Fac. des Sci. de Toulouse*, vol. 9, no. 1.

where (ξ, η, ζ) are the coördinates of the point corresponding to (x, y, z). Consider a set of these corresponding surfaces, one from each system; the conjugate system on the first, whose correspondent on the second is a conjugate system, corresponds to the double system of asymptotic lines on its associate.* Hence the triple system of associate surfaces is composed of surfaces cutting mutually in their asymptotic lines. But such a system is impossible; hence

No two triply conjugate systems of surfaces can correspond by orthogonality of linear elements.

Since, in the infinitesimal deformation of a surface that conjugate system of lines is preserved which has a conjugate system for its correspondent on a surface corresponding by orthogonality of linear elements, it follows from the above result that

A triply conjugate system of surfaces cannot be deformed infinitesimally into another triply conjugate system.

Suppose there is given a triple system of minimal surfaces cutting one another along the conjugate lines of length zero for each surface. Corresponding to the three surfaces determining a point we have three other surfaces—the *adjoints* of the former—determining the corresponding point in the system of adjoint surfaces. Moreover, it is evident that the three surfaces of this latter system cut one another in the conjugate lines of length zero. Since a minimal surface and its adjoint correspond by orthogonality of linear elements, this second triply conjugate system of mimimal surfaces would have this kind of correspondence with the first system. Since this is impossible, we have

There cannot be a triply conjugate system of minimal surfaces cutting one another in the conjugate lines of length zero.

Again suppose that there can be obtained a triply orthogonal system of minimal surfaces. By the Dupin-Darboux theorem we have that these surfaces intersect one another in lines of curvature. In connection with this system consider the triple system of adjoint minimal surfaces of the given system. Since the lines of curvature on a minimal surface correspond to the asymptotic lines on its adjoint, the surfaces of the second system will cut one another in asymptotic lines. Hence the hypothesis was incorrect, and therefore

There cannot be a triply orthogonal system of minimal surfaces.

PRINCETON, N. J.,
December 18, 1900.

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* Bianchi, Lezioni, p. 279.

SHORTER NOTICES.

A Short History of Astronomy. (The University Series.) By ARTHUR BERRY. New York, Scribner, 1899. xxxi + 440 pp.

IT is a sufficiently difficult and thankless task to write a history of any science at the present time, and the difficulty is all the greater when the subject is as old as astronomy. Anyone engaged in a special line of research will always feel some interest in the history of his study quite apart from the value it may have for its advancement, but most of the sciences are of such a nature that only a slight acquaintance with their earlier history is necessary. Astronomy stands almost alone in one respect. Though, as with other sciences, its history is that of discarded *theories*, yet the *observations* made in the past, instead of being less valuable according as they were more ancient, are frequently more so, and the necessity for examining carefully the records from the earliest times has become increasingly important. A well recorded eclipse, passage of a comet, or rough measure of a star's position, may help us more in the determination of certain constants than the most careful series of observations made in recent times.

In the present volume Mr. Berry has made the first attempt to give a continuous and concise popular account of the history of astronomy from the earliest times up to the present day. He has endeavored to make it readable, to set forth the main facts in their due proportion without too great a sacrifice of details, to give sufficient indications of the literature so that one can obtain more information on special points if desired, and finally, to go as far as is reasonable into explanations so that the true nature of a discovery may be appreciated. He has, however, not been contented to make a mere compilation from the earlier works on separate parts of the history but has given a careful and well-written account of the whole subject as far as was possible within the limits set. Moreover, he has, in many cases, gone to the sources (*e. g.*, for the work of Galilei, Kepler) and studied the writings of the pioneers in the original, so that his estimates of what they achieved will have an independent value. The whole book bears evidence of a large amount of work and in spite of its modest title should take a higher rank than many of the popular volumes which appear so frequently at the present time.

The first three chapters are devoted to astronomy from the earliest times to the beginning of the sixteenth century. The author has confined his account almost entirely to the East, omitting the astronomy of the Egyptians, the Chaldeans and the Chinese, owing partly to the difficulty of obtaining first-hand information and partly to the unsettled state of the interpretations which Oriental scholars put on the records which have come down to us. We hope that this omitted portion may in the future be described by as able a hand. Whole chapters are devoted to Copernicus, Tycho Brahe, Galilei, and Kepler, showing how the subject was gradually emancipated from mere speculation and emerged into a science in which observations were collected and theories tested by means of them.

Of the discoveries of Newton and his immediate successors, Mr. Berry naturally gives a full account. From this time, the subject has been divided into two parts, gravitational and descriptive. The former may be said to consist almost entirely in the complete verification of the law of the inverse square coupled with the laws of motion. In later times, the latter has been subdivided into observational, physical, and descriptive astronomy, and the author finds it necessary to adopt these subdivisions for the eighteenth and nineteenth centuries. A full chapter is devoted to Herschel as the first man to systematize observational astronomy. The description of the work of the last century is necessarily much condensed as a large number of partly isolated subjects have to be discussed; but, wherever possible, the main idea involved in every advance has been given.

ERNEST W. BROWN.

Die Mathematiker und Astronomen der Araber und ihre Werke. (Abhandlungen zur Geschichte der mathematische Wissenschaften mit Einschluss ihrer Anwendungen. X Heft.) Von HEINRICH SUTER. Leipzig, B. G. Teubner, 1900, ix + 278 pp.

A CATALOGUE of over five hundred names of mathematicians and astronomers does not give promise of much interest to a reader. But anyone who takes the trouble to dip into Dr. Suter's volume will be pleasantly disappointed if he merely expects to find a list of authors and writers with the titles of their works. To read through such a book is of course out of the question, even for a reviewer, and yet merely to put it on the shelf ready for a chance reference would be doing it an injustice. Although the author has kept in mind his main idea, which is to give an outline of the work done by the Arabians and where it may be found, he has inserted many interesting details about the lives of those who labored for science during the period under consideration.

For example, a choice morsel from the autobiography of Ibn Sînâ (born 980 A. D.), who certainly did not suffer from false modesty, runs as follows: "* * * Under him (one of his teachers, el-Nâtalî) I began the Isagogus; but when it was known that I was obliged to correct him in the definitions of the categories, every one was filled with amazement. Whatever the question put to me, I solved it better than he did and when we turned to the study of logic I showed him that he did not understand the more difficult parts of it. * * * I went on then to the Almagest. When I got to the end of the introductory propositions and arrived at the geometrical ones, el-Nâtalî told me to go and read the book alone and then he and I would read it together and he would point out which results were correct and which incorrect. I began to study it and it soon appeared that I had to explain many of the propositions since he had not, up till then, properly understood them. * * *" However, Ibn Sînâ was somewhat of a *bon vivant* and, though he wrote several works, left no great mark on his subject.

Here and there occur names that bring back pleasant memories of the Arabian Nights, a world of fiction which seems far distant from our subject. But in those days the progress of any kind of intellectual culture was largely in the hands of the reigning prince. The astronomer expected to earn his living by making out horoscopes and casting nativities, whether he believed in them or not, just as surely as the mathematician of our own times expects to make his by teaching; and the success which might attend him probably depended much more on his luck and the ability of the ruler than on his capacity as an astronomer.

The period selected by Dr. Suter runs from 750 A. D. to 1600 A. D. which includes, according to him, everyone worth a notice. For the first hundred and fifty years of this time the Arabians were gathering together and assimilating the works of those who had gone before, chiefly from the schools of Greece and India. A succession of powerful chalifs, amongst whom we notice our old friend Hârûn-el-Rašîd, had been the means of stimulating intellectual progress along with the general rise of civilization amongst the people. They gathered around the court the ablest men and encouraged the formation of schools. The best work was done during the next 375 years, and it was in that time that such additions as the Arabians were able to make to the subject were achieved. From 1275 on, the scientific men were chiefly employed in designing new kinds of astrolabes and quadrants and in the writing of text-books.

Interesting as Arabian science may be to the general historian it is not of very great importance to the mathematician. But the Arabians did a great service in keeping alive the works of their predecessors during a dark period in Europe. Many of the discoveries of the Greeks have been preserved in the numerous manuscripts which the Arabians left and which have escaped destruction. And not the least of their contributions was the introduction of our present number system which they obtained from India and used in preference to the cumbrous Greek and Roman methods. If for no other reason, they deserve to be remembered for this.

ERNEST W. BROWN.

NOTES.

THE eighth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Cornell University, Ithaca, N. Y., beginning Monday, August 19, 1901. Members of the Society will thus be enabled to attend both the summer meeting and the meeting of the American association for the advancement of science which is held at Denver, Col., beginning August 24. In connection with the summer meeting of the Society, arrangements are being made for a colloquium. Professor E. W. BROWN has consented to give a course of six lectures on "Modern methods of dealing with the problems of dynamics and especially those of celestial mechanics, consisting mainly of the work of Poincaré in this direction." The precise title of these lectures will be announced later. A second course of lectures will also be arranged.

AT a meeting of the Cambridge philosophical society, held on November 26, 1900, the following mathematical papers were read: By Mr. T. J. I'A. BROMWICH: "Some theorems in regard to matrices."—By Mr. J. H. GRACE: "On the rational space curve of the fourth order."

THE Cracow academy of sciences, at its general meeting on May 18, 1900, awarded to Dr. G. A. Miller, of Cornell University, the prize, amounting to two hundred and sixty dollars, offered by the academy for the treatment of the following subject:

"To find all the groups of a system of ten letters; or at least to increase the number of the known groups of this degree. (Prize, three hundred florins, with interest from May 15, 1886.)"

This would appear to be the first prize in pure mathematics awarded by a foreign academy to an American.

PARIS ACADEMY OF SCIENCES.—At the regular meeting of the academy, December 10, 1900, M. PAUL PAINLEVÉ, of the École normale, was elected to membership in the section of geometry to the portfolio of Professor G. Darboux who recently became perpetual secretary for the mathematical sciences as successor to the late Joseph Bertrand. The other candidates nominated by the section of geometry were MM. GOURSAT and HUMBERT in the second line, and MM. BOREL and HADAMARD in third line.

At the annual meeting of the academy held December 17, 1900, the following prizes, among others, were awarded: Grand prize of the mathematical sciences to M. MATHIAS LERCH; the Francoeur prize to M. EDMOND MAILLET; the Poncelet prize to M. LÉON LECORNU; the Montyon prize (in mechanics) to M. LEROSEY; the Lalande prize to M. GIACOBINI; and the Janssen prize to M. BARNARD. The Bordin prize was not awarded.

THE capital of the fund for the SYLVESTER medal, held in trust by the London royal society (see BULLETIN, volume 4 (1898), p. 239) now amounts to about four thousand dollars. The following announcement is made in the Royal society yearbook for 1900: The Sylvester medal, which is accompanied by a grant of the balance of the income of the Sylvester medal fund, is awarded triennially for the encouragement of pure mathematical research, irrespective of nationality.

A FUND has been presented to Johns Hopkins University for the annual award of a mathematical prize, to bear the name of SYLVESTER, professor at the University from 1876 to 1883. The prize will consist of a bronze medallion bearing the likeness of Sylvester.

THE University of Kazan has awarded its Lobachevsky prize, consisting of a medal and two thousand roubles, to Professor W. KILLING of Münster for excellent work in the field of mathematics.

VOLUME VIII of C. F. Gauss's Werke has just appeared. It contains papers and fragments on arithmetic, higher analysis, the theory of probability, and geometry. Three other volumes are in preparation. Volume VII will contain a reprint of the Theoria motus and a complete exposition of Gauss's extensive work on the calculation of perturbations. Volume IX will be devoted to mathematical physics and geodesy, especially Gauss's work in connection with the triangulation of Hannover. In Volume X it is intended to collect all biographical material, including extracts from Gauss's correspondence. B. G. Teubner, of Leipzig, is the publisher. (Compare BULLETIN, volume 4 (1898), p. 554.)

ONE of the most recent series of mathematical text-books is the "Sammlung Schubert," which is being published by G. H. Göschen of Leipsic under the direction of Professor H. SCHUBERT of Hamburg. Of the forty volumes of the series twelve have thus far appeared. The latest is by Professor L. SCHLESINGER, Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variablen. The German mathematical publishing houses are beginning to adopt the English plan of issuing their books bound in cloth. This is done in the "Sammlung Schubert," in Teubner's new series, and in some of Veit and Company's new books.

GRUNERT's *Archiv fur Mathematik und Physik* will henceforth be published by B. G. Teubner, of Leipzig. The editors will be Professors E. LAMPE, of Berlin, W. FRANZ MEYER, of Königsberg, and Dr. E. JAHNKE, of Berlin.

COLLEGE OF FRANCE.—The following courses in mathematics and mathematical physics are announced for the first semester of the present academic year, the lectures beginning December 3, and each course consisting of two lectures per week: By Professor M. BRILLOUIN: A critical exposition of some recent attempts at an electro-mechanical theory of voltaic, electro-dynamic, and optical phenomena, after the works of Wiechert, Riecke, Drude, Lorentz, Larmor and others.—By Professor J. HADAMARD: Partial differential equations of mechanics.—By Professor C. JORDAN: Construction of solvable groups.—By Professor E. MASCART: The relations between light and electricity. —By M. É. BOREL: The theory of series with positive terms and integrals with positive elements.

M. HERMANN, of Paris, has issued a new catalogue, No. 69, containing some twenty-three hundred titles of works in mathematics, physics, and astronomy.

CERTAIN of the several scientific posts held by the late Professor Joseph Bertrand have been filled as follows: M. BERTHELOT was elected to the vacant chair in the French academy; Professor G. DARBOUX became perpetual secretary of the Paris academy of sciences; M. BRILLOUIN has been appointed professor of mathematical physics at the College of France; and Professor H. POINCARÉ has been made a member of the council of the Observatory of Paris.

Nature for November 29 contains an abridgment of the address delivered at the opening meeting of the Society of arts on November 21 by Sir JOHN EVANS, entitled "The origin and progress of the scientific societies of Great Britain."

PROFESSOR CELORIA, heretofore assistant astronomer at the observatory of Milan has been appointed director. He succeeds Professor SCHIAPARELLI, who retired from the directorship on November first. Professor Schiaparelli has been connected with the observatory at Milan for the last forty years.

PROFESSOR H. POINCARÉ has been elected a foreign member of the Munich academy of sciences.

PROFESSOR ADOLF KNESER of Dorpat has accepted a call to the Berlin School of Mines.

PROFESSOR G. BAUER of Munich will not lecture again after the close of the present winter semester.

MR. R. W. H. T. HUDSON has been appointed to a fellowship in mathematics at St. John's College of Cambridge University. Mr. Hudson was senior wrangler in 1898, Smith's prizeman of 1900 and is the son of Professor Hudson of King's College, London.

MR. J. A. MCCLELLAND, of Cambridge University, has been appointed to the chair of natural philosophy in University College, Dublin, made vacant by the recent death of Professor Thomas Preston.

PROFESSOR A. G. WEBSTER, of Clark University, has been promoted to a full professorship of physics.

PROFESSOR W. W. CAMPBELL has been appointed director of the Lick observatory, succeeding the late Professor J. E. Keeler.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Archibald (R. C.). The cardioid and some of its related curves. Strassburg, 1900. 4to. 38 pp., 3 plates. M. 3.00

Blake (E. M.). Two plane movements generating quartic scrolls. (*Transactions of the American Mathematical Society*, Vol. I, pp. 421–429.) 4to.

Budisavljević (E. von). Leitfaden für den Unterricht der höheren Mathematik an der k. und k. Artillerie- und der Pionnier-Kadettenschule. Wien, Seidel, 1900. 8vo. 11 + 618 pp. M. 14.40

Carrone (C.). Sopra un nuovo metodo di generazione del complesso tetraedrale. Catania, 1900. 8vo. 16 pp. Fr. 0.75

Dziobek (O.). Lehrbuch der analytischen Geometrie. Teil I: Analytische Geometrie der Ebene. Berlin, Hoffmann, 1900. 8vo. 8 + 350 pp. M. 6.00

Gauss (C. F.). Werke, herausgegeben von der kgl. Gesellschaft der Wissenschaften zu Göttingen. Vol VIII: Nachträge zur Arithmetik, Analysis, Wahrscheinlichkeitsrechnung und Geometrie. Leipzig, Teubner, 1900. 4to. 3 + 458 pp. Boards. M. 24.00

Graham (J.). An elementary treatise on calculus for engineering students, with numerous examples and problems worked out. 2d edition. London, Spon, 1900. 12mo. 288 pp. 7s. 6d.

Guccia (G. B.). Memorie di geometria. Palermo, 1900. 8vo. Fr. 15.00

Guradze (H.). Die Reye'sche Geometrie der Mannigfaltigkeiten projektiver Grundgebilde behandelt mittels einer besonderen Art bilinearer Formen. (Diss.) Breslau, 1900. 8vo. 68 pp.

Hentschel (O.). Ausführung einiger conformer Abbildungen. Teil I. Salzwedel, 1899. 4to. 8 pp., 4 plates. M. 4.00

Herff (E.). Die Maxima und Minima einer verwandelbaren Funktion. Sigmaringen, 1899. 4to. 18 pp.

Hossenfelder (E). Zur Theorie der trigonometrischen Reihe. (Progr.) Strassburg, 1900. 4to. 22 pp.

Kinn (G. A.) Die Anwendung unendlicher Produkte in der Funktionentheorie. Sächsisch-Regen, 1899. 4to. 32 pp.

Netto (E.). Ueber die Grundlagen und die Anwendungen der Mathematik. Giessen, 1900. 4to. 15 pp.

Sachs (J.). Lehrbuch der projektivischen (neueren) Geometrie (synthetische Geometrie, Geometrie der Lage). Teil I: Elemente und Grundgebilde; Projektivität; Dualität. Nebst einer Sammlung gelöster und ungelöster Aufgaben, mit den Ergebnissen der ungelösten Aufgaben. Für das Selbststudium und zum Gebrauche an Lehranstalten bearbeitet. Stuttgart, Maier, 1900. 8vo. 4+182 pp. (Kleyer's Encyklopädie der gesamten mathematischen, technischen und exakten Naturwissenschaften). M. 5.00

SCHEFFERS (G.). Anwendung der Differential- und Integralrechnung auf Geometrie. (In 2 Banden.) Vol. I: Einfuhrung in die Theorie der Kurven in der Ebene und im Raume. Leipzig, Veit, 1901. 8vo. 10 + 360 pp. Cloth. M. 10.00

SCHOLZ (P.). Ueber die Reduktion des Dreikörper-Problems auf die Integration einer einzigen Differentialgleichung. (Diss.) Berlin, 1900. 8vo. 38 pp. M. 1.50

STÖRMER (C.). Sur une propriété arithmétique des logarithmes des nombres algébriques. (*Bulletin de la Société Mathématique de France*, Vol. 28, pp. 1–12.) 8vo.

WEILL (A.). Die geometrische Interpretation der Gleichung fünften Grades auf invariantentheoretischer Grundlage. Strassburg, 1900. 8vo. 60 pp., 3 plates. M. 3.00

WEINER (F.). Eine Anwendung der Hermite'schen U-Functionen. Wien, 1899. 8vo. 16 pp.

II. ELEMENTARY MATHEMATICS.

ALASIA (C.). La recente geometria del triangolo. Città di Castello, 1900. 8vo. 58 pp. Fr. 2.00

BOS (H.). Géométrie élémentaire, rédigée conformément aux programmes officiels de l'enseignement classique pour les classes de quatrième, troisième, seconde, et philosophie. 16e édition. Paris, Hachette, 1900. 16mo. 288 pp. Fr. 2.00

DAUZAT. Eléments de méthodologie mathématique, à l'usage de tous ceux qui s'occupent de mathématiques élémentaires. Paris, Nony, 1901. 8vo. 7 + 102 pp.

DELBRIDGE (C. L.). Delbridge's complete calculator. Vols. 2 and 3. St. Louis, Mo., The Delbridge Co., 1900. Folio. $10.00

DIEKMANN (J.). See HEILERMANN (H.).

DUBOUIS (E.). Le théorème de Mascheroni. Le compas remplace entièrement la règle et l'équerre dans les constructions géométriques Démonstration élémentaire très courte. Vannes, Imprimerie du commerce, 1900. 16mo. 8 pp.

FAJON (H.). Compléments d'algèbre élémentaire. Variations des fonctions du premier degré, du second degré et bicarrées. Paris, 1900. 8vo. Fr. 1.80

F. (F.). Géométrie; cours élémentaire Paris, Poussielgue [1900]. 16mo. 122 pp. (Collection d'ouvrages classiques rédigés en cours gradués, conformément aux programmes officiels.)

——. Géométrie; cours moyen. Paris, Poussielgue [1900]. 12mo 191 pp. (Collection d'ouvrages classiques rédigés en cours gradués, conformément aux programmes officiels.)

GLINZER (E.). Kurzes Lehrbuch der ebenen Trigonometrie fur gewerbliche Schulen sowie zum Selbstunterricht. 2te Auflage. Mit 46 Figuren und vielen Aufgaben. Dresden, Kuhtmann, 1900. 8vo. 7 + 79 pp. M. 0.90

GRÉVY (A.). Algèbre, à l'usage des élèves des classes de troisième, seconde et rhétorique. 2e édition. Paris, Nony, 1900. 18mo. 248 pp. Fr. 2.00

GUNDELFINGER (S.). Sechsstellige Gaussische und siebenstellige gemeine Logarithmen. Leipzig, Veit, 1900. 4to. 4 + 31 pp Cloth. M. 2.80

HEILERMANN (H.) und DIEKMANN (J.). Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Teil II: Die Gleichungen zweiten Grades mit mehreren Unbekannten; die Progressionen; die kubischen und biquadratischen Gleichungen; niedere Analysis. 5te Auflage. Essen, Baedeker, 1900. 8vo. 7 + 238 pp. Cloth. M. 2.80

HUMBERT (E.). Traité d'arithmétique, à l'usage des élèves de mathématiques élémentaires, des aspirants au baccalauréat de l'enseignement classique (deuxième série), et au baccalauréat de l'enseignement moderne (troisième série), et des candidats à l'Institut agronomique, avec des compléments destinés aux candidats aux grandes écoles du gouvernement. Avec une préface de J. Tannery. 2e édition. Paris, Nony, 1901. 8vo. 7 + 483 pp. Fr. 5.00

JAHNKE (E.). Ueber dreifach perspektivische Dreiecke in der Dreiecksgeometrie. (Progr.) Berlin, 1900. 4to. 26 pp.

LECHTHALER (A.). Zur Lehre von den geometrischen Verhältnissen und Proportionen, und der allgemeine Proportionalitätssatz. (Progr.) Linz, 1900. 8vo. 23 pp.

LÜBSEN (H B.). Ausführliches Lehrbuch der Arithmetik und Algebra zum Selbstunterricht und mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. 25ste Auflage. Leipzig, Brandstetter, 1900. 8vo. 6 + 261 pp. M. 4.00

MARTUS (H C. E.). Mathematische Aufgaben zum Gebrauche in den obersten Klassen höheren Lehranstalten, aus den bei Reifeprufungen an deutschen höheren Schulen gestellten Aufgaben ausgewahlt und mit Hinzufügung der Ergebnisse (4ter Teil) zu einem Uebungsbuche vereint. Teil 3: Aufgaben. Dresden, Koch, 1900. 8vo. 8 + 172 pp. M. 4.00

SALOMON (MME. A.). Leçons d'arithmétique, à l'usage de l'enseignement secondaire des jeunes filles (classes de deuxième et troisième années). 2e édition. Paris, Nony, 1900. 18mo. 8 + 217 pp. Fr. 2.00

SCHWERING (K.). Anfangsgründe der Trigonometrie für die sechste Stufe höherer Lehranstalten, nach den amtlichen Lehrvorschriften bearbeitet. 2te Auflage. Freiburg i. B., Herder, 1900. 8vo. 12 pp. (From "Schwering, Trigonometrie für höhere Lehranstalten.") M. 0.20

TANNERY (J.). See HUMBERT (E.).

III. APPLIED MATHEMATICS.

ARAGO. See MEMOIRS.

ARRHENIUS (S.). Lärobok i teoretisk electrokemi. Stockholm, 1900. 8vo. 266 pp. M. 5.00

AUTENRIETH (E.). Technische Mechanik. Ein Lehrbuch der Statik und Dynamik fur Maschinen- und Bauingenieure. Berlin, Springer, 1900. 8vo. 22 + 558 pp. M. 12.00

Bigelow (F. H.). Report on the international cloud observations, May 1, 1896, to July 1, 1897. Prepared under the direction of W. L. Moore, Chief of the Weather Bureau. Washington, Government Printing Office, 1900. 4to. 787 pp. (Vol. II. of the Report of the Chief of the Weather Bureau for 1898–99). Cloth.

Blythe (W. H.). Geometrical Drawing. With notes and examples. Part II: Solid or descriptive geometry. Cambridge series for schools and training colleges. 16mo. Cloth. Pp. xi + 193–328. $0.50

Bolte (F.). Die Nautik in elementarer Behandlung. Einführung in die Schiffartkunde. Mit 90 vollständig gelösten Beispielen, 260 analogen ungelösten Aufgaben mit den Ergebnissen, nebst 88 Figuren, sowie Erklärung der Kunstausdrücke der Seemannssprache. Stuttgart, Maier, 1900. 8vo. 8 + 196 pp. M. 5.00

Bürklen (O.). Graphisches Rechnen und graphische Darstellungen im Mathematik-Unterrichte. Gmünd, 1899. 8vo. 58 pp.

Crew (H.). See Memoirs.

Ebert (H.). Die Theorie des Elektromagnetismus. Leipzig, 1900. 8vo. 8 + 83 pp. M. 4.50

Eggers (W.). Lehrbuch der darstellenden Geometrie. Teil II: Durchschnitte und Durchdringungen der Körper. Leipzig, 1900. 8vo. 8 plates. M. 1.50

Fresnel (A. J.). See Memoirs.

Hefft (O.). Scheinbare Anziehung und Abstossung von Kugeln, die in einer klebrigen Flüssigkeit rotieren. (Diss.) Heidelberg, 1900. 8vo. 47 pp., 10 plates.

Hirszberg (L.). Ueber elektrische Potentialmessungen bei Spitzenentladungen in Luft von normalem Druck. (Diss.) Heidelberg, 1900. 8vo. 29 pp., 4 plates.

Huygens (C.). See Memoirs.

Junge (K.) Ueber die magnetischen Kraftlinien Faradays. (Progr.) Leipzig, 1900. 4to. 32 pp.

Kapp (G.). Elektrische Wechselströme. Deutsch von H. Kaufmann. 3te Auflage. Leipzig, Leiner, 1900. 8vo. 5 + 92 pp. M. 2.00

Kaufmann (H.). See Kapp (G.).

Kohlrausch (F.). Die Energie oder Arbeit und die Anwendungen des elektrischen Stromes. Leipzig, Duncker & Humblot, 1900. 8vo. 5 + 77 pp. M. 2.40

Langer (H.). Ueber die mechanischen Prinzipien der modernen Elektrizitätslehre. Fortsetzung. (Progr.) Linz, 1900. 8vo. 44 pp.

Memoirs on the wave theory of light, edited by H. Crew. Contents: C. Huygens, Treatise on light; Biographical sketch of Huygens; T. Young, On the theory of light and colors; T. Young, An account of some cases of the production of colors not hitherto described; T. Young, Experiments and calculations relative to physical optics; Biographical sketch of Young; A. J. Fresnel, Memoir on the diffraction of light; Arago and Fresnel, On the action of rays of polarized light upon each other; Biographical sketch of Fresnel; Bibliography on the wave theory of light. New York, American Book Company [1900]. 8vo. 15 + 164 pp. (Scientific memoirs, No. 10.) Cloth. $1.00

MONDIET (O.) et THABOURIN (V.). Traité élémentaire des mécanismes, à l'usage des élèves des établissements d'enseignement moderne, des écoles industrielles et des écoles d'arts et métiers. 4e édition. Paris, Hachette, 1901. 8vo. 192 pp.

NEUHOFF (O.) Adiabatische Zustandsänderungen feuchter Luft und deren rechnerische und graphische Bestimmung. (Diss.) Berlin, 1900. Folio. 33 pp., 1 plate. M. 2.00

SCHMIDT-ULM (G.). Die Wirkungsweise, Berechnung und Konstruktion der Gleichstrom-Dynamomaschinen und Motoren. 2te Auflage. Leipzig, Leiner, 1900. 8vo. 7 + 264 pp., 42 plates. M. 8.50

TATE (W. G.). Theory and practice of navigation and nautical astronomy. London, Potts, 1900 8vo. 122 pp. Cloth. 6s.

THABOURIN (V.). See MONDIET (O.).

WEINSTEIN (B.) Die Erdströme im deutschen Reichstelegraphengebiet und ihr Zusammenhang mit den erdmagnetischen Erscheinungen. Mit einem Atlas enthaltend 19 lithographirte Tafeln in 4to. Braunschweig, Vieweg, 1900. 8vo. 7 + 78 pp. M. 4.00

YOUNG (T.) See MEMOIRS.

Book No............
LIBRARY.

THE SEVENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Seventh Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Friday, December 28, 1900. An important feature of the occasion was the election of officers and other members of the Council, at which recently adopted amendments of the Constitution went into effect. The term of office of the President, who must have served previously, but with a year's interval, as Vice-President, is now increased to two years. Both Presidents and Vice-Presidents are made ineligible for immediate reëlection. Four members of the Council, instead of three, are now elected annually. This provision and the permanent membership of the ex-Presidents increases the number of seats in the Council to twenty-four at present. The growth of the Society has been such that the numerical basis of representation of the Council is almost precisely the same as that of six years ago, when the Society was reorganized as a national body.

At the close of the century a brief recapitulation of the advance of the Society during these six years is not inappropriate. The record reveals in the past a happy and substantial augury for the future. The membership has grown from 244 in September, 1894. to 357 on January 1, 1901. In 1894 the number of papers read at the meetings was 24; in 1900 it was 112. This expansion in numbers and output has had for one valuable result the creation, in 1898, of the Chicago Section of the Society, an event which has proved not only a great benefit to many members to whom the New York meetings are practically inaccessible, but a source of strength to the Society as a whole. The demand for improved facilities for publication, occasioned by the great increase in productive activity of the members, has had for its outcome the founding of the *Transactions*, which has now successfully completed its first year of existence as the official organ of the Society for the publication of important original papers read before it. Meanwhile the BULLETIN has been considerably increased in size, although confining itself more strictly than before to the historical and critical field for which it was originally designed. A gratifying recognition of the usefulness and efficiency of the Society is evidenced by the liberal financial coöperation of ten leading universities of the country in the publication of the

Transactions. Since 1894 summer meetings, of great interest and largely attended, have been held, usually in connection with large and general scientific gatherings. On two occasions the summer meeting has been supplemented by colloquia, or special courses of lectures summarizing important fields of recent investigation. The attendance at the meetings throughout the year has constantly increased. In 1900 the total attendance was 209, and the number of members attending at least one meeting during the year was 110. The administration of business has been wisely left from the beginning in the hands of the Council, the time of the meetings being thus economized for purely scientific purposes. The finances of the Society are in a satisfactory condition, and the extra expense of publishing the *Transactions* has been met without a reduction of the small surplus which has slowly accumulated in the course of years. Over one-third of this surplus consists, however, of life membership payments, which have been set apart as a separate invested fund to be devoted to such purposes as may hereafter be approved. The Society is accumulating from the exchanges of the BULLETIN and the *Transactions* and from numerous gifts a considerable library of mathematical journals and works. It is to be hoped that this collection may be bound and made accessible to the members generally within a short time. The Society, however, has at present no funds available for this purpose.

The attendance at the annual meeting numbered about forty persons, including the following thirty-three members of the Society :—

Dr C. L. Bouton, Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. W. B. Ford, Mr. A. S. Gale, Professor Harris Hancock, Dr. G. W. Hill, Dr. A. A. Himowich, Dr. J. I. Hutchinson, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. Emory McClintock, Dr. James Maclay, Professor Frank Morley, Dr. D. A. Murray, Professor W. F. Osgood, Mr. J. C. Pfister, Professor M. I. Pupin, Professor J. K. Rees, Professor C. A. Scott, Professor P. F. Smith, Dr. Virgil Snyder, Dr. W. M. Strong, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor A. G. Webster, Professor L. A. Wait, Mr. E. B. Wilson, Professor F. S. Woods.

The meeting extended as usual through a morning and an afternoon session. Vice-President Thomas S. Fiske occupied the chair. The Council announced the election of the following persons to membership in the Society :—Dr. G. N.

Bauer, University of Minnesota, Minneapolis, Minn.; Dr. J. R. Benton, Princeton University, Princeton, N. J.; Dr. H. F. Blichfeldt, Leland Stanford University, Palo Alto, Cal.; Dr. G. A. Bliss, University of Minnesota, Minneapolis, Minn.; Professor Joseph Bowden, Adelphi College, Brooklyn, N. Y.; Professor D. F. Campbell, Armour Institute, Chicago, Ill.; Dr. J. E. Clarke, Gilbert School, Winsted, Conn.; Dr. Euplio Conoscente, New York, N. Y.; Mr. Arthur Crathorne, University of Wisconsin, Madison, Wis.; Mr. H. W. Curjel, Birkdale, Lancashire, Eng.; Professor L. M. Defoe, University of Missouri, Columbia, Mo.; Mr. B. S. Easton, University of Pennsylvania, Philadelphia, Pa.; Dr. L. P. Eisenhart, Princeton University, Princeton, N. J.; Mr. U. S. Hanna, University of Indiana, Bloomington, Ind.; Mr. L. I. Hewes, Yale University, New Haven, Conn.; Mr. A. M. Kenyon, Purdue University, Lafayette, Ind.; Professor C. N. Little, Leland Stanford University, Palo Alto, Cal.; Mr. E. L. Milne, University of Illinois, Champaign, Ill.; Mr. H. B. Mitchell, Columbia University, New York, N. Y.; Dr. Asutosh Mukhopâdyây, Calcutta, India; Mr. W. S. Nichols, New York, N. Y.; Professor J. M. Page, University of Virginia, Charlottesville, Va.; Dr. T. H. Taliaferro, Pennsylvania State College, State College, Pa.; Miss R. H. Vivian, University of Pennsylvania, Philadelphia, Pa., Mr. N. R. Wilson, Royal Military Academy, Kingston, Canada. Six applications for membership in the Society were reported. Reports were received from the Secretary, Treasurer, Librarian, and the Auditing Committee. These reports have been printed in the Annual Register, just issued.

At the annual election the following officers and members of the Council were chosen:

President,	Professor Eliakim H. Moore.
Vice-Presidents,	Professor T. S. Fiske, Professor H. S. White.
Secretary,	Professor F. N. Cole.
Treasurer,	Dr. W. S. Dennett.
Librarian,	Professor Pomeroy Ladue.

Committee of Publication,

Professor F. N. Cole,
Professor Alexander Ziwet,
Professor Frank Morley.

Members of the Council to serve until December, 1903,

Professor E. W. BROWN,
Professor H. B. FINE,
Professor T. F. HOLGATE,
Professor W. F. OSGOOD.

Member of the Council to serve until December, 1902,

Professor E. W. HYDE.

The following papers were presented at this meeting:

(1) Dr. VIRGIL SNYDER: "On some plane curves having factorable parallels."

(2) Professor E. D. ROE: "On a formula of interpolation."

(3) Mr. W. B. FORD: "Dini's method of showing the convergence of Fourier's series and of other allied developments."

(4) Dr. EMORY MCCLINTOCK: "A simplified solution of the cubic."

(5) Professor W. F. OSGOOD: "On the existence of a minimum of the integral

$$\int_{x_0}^{x_1} F(x, y, y')dx$$

when x_0 and x_1 are conjugate points, and the geodetics on an ellipsoid of revolution;" a revision of a theorem of Kneser.

(6) Mr. C. J. KEYSER: "Theorems concerning positive definitions of finite assemblage and infinite assemblage."

(7) Professor M. I. PUPIN: "Wave propagation over bridged wave conductors."

(8) Professor F. MORLEY: "On a point in Sylvester's theory of canonical forms."

(9) Professor HARRIS HANCOCK: "On primary prime functions in several variables and a generalization of an important theorem of Dedekind."

(10) Dr. J. I. HUTCHINSON: "On some birational transformations of a Kummer surface into itself."

(11) Miss R. G. WOOD: "The collineations of space which transform a non degenerate quadric surface into itself."

(12) Professor H. E. SLAUGHT: "The complete form system of invariants of the group of 120 quadratic Cremona transformations of the plane."

(13) Dr. JAMES MACLAY: "Some geometrical theorems conected with a class of differential equations derived from Poisson's equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left(\frac{\partial^2 z}{\partial x^2} - \frac{mz}{x^2} \right)."$$

(14) Mr. J. K. WHITTEMORE: "Cones of the second degree osculating developable surfaces"

(15) Professor E. O. LOVETT: "The types of line-sphere transformation."

(16) Professor E. O. LOVETT: "Note on differential geometry of n-dimensional space."

(17) Dr. L P. EISENHART: "A demonstration of the impossibility of a triply asymptotic system of surfaces."

(18) Professor MAXIME BÔCHER: "Some cases in which the identical vanishing of the Wronskian is a sufficient condition for linear dependence."

In the absence of the authors, Mr. Whittemore's paper was read by Professor Osgood, and the papers of Professor Roe, Professor Slaught, Professor Lovett, Dr. Eisenhart, and Professor Bôcher were read by title. Dr. Eisenhart's paper appeared in the January number of the BULLETIN. The papers of Mr. Ford, Mr. Keyser, and Dr. Hutchinson are included in the present number. Abstracts of the other papers are given below.

Dr. Snyder's paper is summarized as follows: Taking a scroll contained in a linear congruence and of the type having factorable asymptotic lines, one can transform the directrices into two anti-parallel planes in sphere space, and the generators into spheres which envelope a tubular surface. The lines of curvature are cut from this surface by a series of parallel planes. These sections are projections of parallels to the locus of centers of the surface, and break up into two factors. The same problem is then solved, starting from a surface of revolution, without using line geometry.

Professor Roe extends the formula

$$f(x + y) = (1 + \Delta)^y f(x),$$

for a positive integral y, to all real values of y within the limits of convergency and for the simplest expansion of the symbol $(1 + \Delta)^y$, by the immediate and direct relation of two series of the same function to each other, and independently of any Taylor series. Both symbolic and non-symbolic proofs are given. The following symbolic proof is given for the extension to a positive fractional commensurable y. Two series of the same function are taken

$$f(x+0)=u_0,\ f(x+1)=u_1,\ \cdots,\ f(x+r)=u_r,\ \cdots$$

with differences Δu's,

$$f(x+0)=v_0,\ f\left(x+\frac{1}{\lambda}\right)=v_1,\ \cdots,\ f\left(x+\frac{\nu}{\lambda}\right)=v_\nu\ \cdots$$

with differences δv's.

ν and λ are integers. It is shown that $v_\nu=(1+\delta)^\nu v_0$; then it is easily found that $\Delta^r u_0=[(1+\delta)^\lambda-1]^r v_0$; hence, with the confined meaning of $(1+\Delta)^{\frac{\nu}{\lambda}}$, that

$$(1+\Delta)^{\nu'}u_0=u_0+\binom{\nu'}{1}\Delta u_0+\binom{\nu'}{2}\Delta^2 u_0+\cdots+\binom{\nu'}{r}\Delta^r u_0+\cdots$$

$$=v_0+\binom{\nu'}{1}[(1+\delta)^\lambda-1]v_0+\binom{\nu'}{2}[(1+\delta)^\lambda-1]^2 v_0$$

$$+\cdots+\binom{\nu'}{r}[(1+\delta)^\lambda-1]^r v_0+\cdots$$

$$(\text{where } \nu'=\nu/\lambda)$$

$$=[1+(1+\delta)^\lambda-1]^{\nu'}v_0=(1+\delta)^{\nu'\lambda}v_0=(1+\delta)^\nu v_0$$

$$=v_\nu=f\left(x+\frac{\nu}{\lambda}\right),\ \text{or}\ f\left(x+\frac{\nu}{\lambda}\right)=(1+\Delta)^{\frac{\nu}{\lambda}}f(x).$$

Extension is then made to an integral negative y, next to a negative fractional commensurable y, and lastly by limits to an incommensurable value of y, thus securing the extension to all real values of y for a convergent expansion.

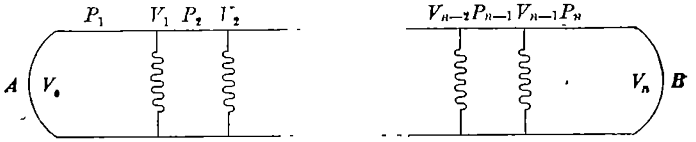
Dr. McClintock's "simplified solution of the cubic" is essentially a rearrangement, in a strikingly simple form, of Bézout's "earlier solution," with certain novel features. It will appear in the *Annals of Mathematics*.

In a paper entitled "Zur Variationsrechnung" (*Mathematische Annalen*, volume 50 (1897), p. 50), Kneser enunciates the theorem that the integral

$$I=\int_{x_0}^{x_1}F(x,y,y')dx$$

ceases to be a minimum, not only when the interval (x_0, x_1) contains in its interior a point x' conjugate to x_0, but when x_1 coincides with x'. This theorem is true in general, but not in all cases, and it is the object of Professor Osgood's paper to point out a general class of cases for which it is not true. The geodetics on an ellipsoid of revolution are a case in point. The paper will appear in the *Transactions*.

In Professor Pupin's paper the wave conductor is the loop AB in the figure. The bridges are denoted by $1, 2, \cdots, n-2, n-1$. They divide the loop into n equal parts. Let l be



the distance between the bridges; L_1, R_1 the inductance and resistance, respectively, of each bridge; L, R, C, the inductance, resistance, and capacity, respectively, per mile, of the wave conductor. Let P be the potential and y the current at any point. P is continuous and y has as many discontinuities as there are bridges. In any section like m we shall denote by P_m and y_m the potential and current at any point of the section. Denoting the distance of any point of this section from the beginning of the section by s, we shall have

$$P_m = V_m, \qquad y_m = x_{2m+1}, \qquad \text{when } s = l;$$

$$P_m = V_{m-1}, \qquad y_m = x_{2m}, \qquad \text{when } s = 0.$$

The problem can now be stated as follows: Find the integral of the differential equation

$$L\frac{d^2P}{dt^2} + R\frac{dP}{dt} = \frac{1}{C}\frac{\partial^2 P}{\partial s^2}$$

which will satisfy n boundary conditions of the type

$$h(x_{2m-1} - x_{2m}) - 2V_m = 0,$$

where $h = ipL_1 + R_1$, and p is the frequency speed of the impressed simple harmonic electromotive force. The solution is

$$V_m = \frac{E}{2}\,\frac{\sin 2(m-n)\theta}{\sin 2n\theta},$$

where

$$\cos 2\theta = \cos \rho l - \frac{\rho \sin \rho l}{ipCh}, \qquad \rho^2 = p(pL - iR),$$

$$P_{m+1} = \frac{V_m \sin \mu(l - s) + V_{m+1} \sin \mu s}{\sin \mu l},$$

$$y_{m+1} = \frac{2ipC}{\mu \sin \mu l} \{ V_{m+1} \cos \mu s - V_m \cos \mu(l - s) \}.$$

When the bridges are sufficiently near with respect to the wave length which corresponds to the frequency speed of the force on the wave conductor without the bridges, then the system acts like a uniform wave conductor the capacity of which depends on the frequency. There is a critical frequency at which the wave length becomes infinite.

Professor Morley's paper is in abstract as follows: In the *Philosophical Magazine* for 1851 Sylvester gave a canonical form for a binary quantic of even order, as far as the eighth order, and suggested a general form which involved the finding of a certain covariant V. It seems to have been supposed that the difficulty of continuing lay in finding V (see Elliott, Algebra of quantics, p. 299). An examination shows that there is no theoretic difficulty, but that the source of V is easily obtainable as a determinant. The real objection to the theory lies in the complication of the expression of V by means of the fundamental covariants. Thus for the quantic of the tenth order, in terms of the covariants of Sylvester's auxiliary quintic, we have, in Elliott's notation for the quintic,

$$\begin{aligned} V = 100\, C_{64} C_{51} - 25\, C_{80} C_{15} + 3\, C_{40}^2 C_{36} + 120\, C_{71} C_{22}^2 \\ + 92\, C_{40} C_{75} + 24\, C_{40} C_{22} C_{53}. \end{aligned}$$

This formula was calculated by Mr. Coble.

Dedekind (in Supplement XI to Dirichlet's Zahlentheorie, 4th edition, p. 570) shows, if ω is any algebraic integer in a fixed realm of rationality Ω, that

$$\omega^{p^f} \equiv \omega \quad (\text{mod. } \mathfrak{p}),$$

where p is a rational prime integer that is divisible (in Ω) by the prime ideal $\mathfrak{p}$ and f is the degree of $\mathfrak{p}$. He then shows, pp. 570–572, that

$$t^{p^f} - t \equiv \Pi(t - \omega) \equiv \Pi P(t) \quad (\text{mod. } \mathfrak{p}),$$

where t is a variable and where the first product is taken over a system of incongruent (mod. $\mathfrak{p}$) algebraic integers in Ω and the second product is taken over all primary prime functions $P(x)$, whose degree is f or a divisor of f. The investigations are restricted to algebraic numbers in a fixed realm. Professor Hancock considers instead of the algebraic numbers integral functions in any number of variables whose coefficients are the Dedekind algebraic numbers; instead of the prime ideal $\mathfrak{p}$ there occur in his general treatment prime modular systems, in whose elements enter all the variables save one. Primary prime functions are taken in which the variable that is wanting in the modular system occurs to degree h and where the coefficients are integral functions of the other variables that are found in the modular system. The number of such primary prime functions whose degree is h or a divisor of h is calculated, and then the generalized Dedekind's theorem is developed.

Miss Wood's paper deals with the collineations of space which transform a non-degenerate quadric into itself. Starting with the theorem that such a collineation which leaves the two systems of generators invariant may be compounded of two skew reflections, the following theorems in non-euclidean geometry are deduced. For elliptic space: 1° If a straight line be displaced, the loci of the middle points of chords joining congruent points are straight lines. 2° Planes perpendicular to these chords at their middle points intersect each set in a straight line. 3° If a plane figure be displaced, the loci of the middle points of chords joining congruent points are planes. 4° Planes perpendicular to these chords at their middle points pass each set through a point. For hyperbolic space: 1° If a straight line be displaced, the locus of the middle point of chords forming congruent points is a straight line. 2° Planes perpendicular to these chords at their middle points intersect in a line. 3° If a plane figure be displaced, the middle points of chords joining congruent points lie in a plane. 4° Planes perpendicular to these chords at their middle points pass through a point in the plane. When the quadric degenerates four well known theorems of Chasles for euclidean space are obtained.

In a paper soon to be published in the *American Journal of Mathematics*, Professor Slaught has discussed the nature of invariants of a quadratic group and has shown that, inasmuch as the degree of any rational integral form is doubled under a quadratic transformation, only a rational fraction can transform into itself after throwing off a common factor in the variables from numerator and denominator. For the quadratic Cremona group of 5! cross ratio transformations it is shown that the most general forms suitable for numerator and denominator of such invariant fractions are of degree $6n$ and throw off a factor $(z_1z_2z_3)^{2n}$ under the quadratic generator

$$z_1' : z_2' : z_3' = z_2z_3 : z_1z_3 : z_1z_2.$$

Such a form, R_{6n} is found to be decomposable into two factors

$$R_{6n} = P^{2\mu} R_{6(n-2\mu)}$$

in which

$$P = z_1z_2z_3(z_1 - z_2)(z_1 - z_3)(z_2 - z_3),$$

$\mu = 0$ or a positive integer, and $R_{6(n-2\mu)}$ contains *no factor* of P.

The factors of P, set equal to zero, are the six sides of the complete quadrangle whose vertices are the four critical points of the group, and every invariant curve

$$R_{6n} = 0$$

has at each of these critical points a multiple point of order $2(n + \mu)$.

The quadratic group G_{120} has a linear subgroup $G^{(1)}_{24}$ which is projectively related to the Klein's linear group of order 4!, and through a study of the invariants of these two linear groups there are found the most general invariant forms under G_{120} of degree 6, 12, and 18 $[n = 1, 2, 3]$, namely,

$$mA, \quad m_1A^2 + m_2P^2, \quad m_1A^3 + m_2AP^2 + m_3C,$$

in which the m's are arbitrary and A, P^2, C are three invariants of the binary quintic form, written in terms of the cross ratios of the five roots of the quintic. Then by a series of theorems relating the general forms R_{6n} to the geometry of the group [The geometric representation was given in the October number of the *American Journal of Mathematics.*] it is shown, through a process of successive reduction, that any invariant form under G_{120} is expressible as a rational integral function of A, P^2, C, and thus a system of fundamental forms is established in terms of which all invariant fractions under the quadratic group can be expressed.

As is known, any three solutions of the differential equation

$$\frac{\partial^2\vartheta}{\partial u\partial v} + f(u, v)\frac{\partial\vartheta}{\partial u} + \varphi(u, v)\frac{\partial\vartheta}{\partial v} = 0,$$

when set equal to x, y, z, respectively, define a surface referred to a system of conjugate lines. Dr. Maclay determines a transformation of a character to change conjugate systems of curves on a surface into conjugate systems on the transformed surface. After some discussion of the relations of a surface and its transformed in the general case, the surfaces defined by

$$f = \varphi = \frac{n}{u + v}$$

and the corresponding transformed surfaces are more particularly studied. By suitable transformations the latter equation is reduced to the form given in the title of the paper, and the complete integral is obtained. The transformation, when employed upon the general surface of translation, will change the curves of translation into lines of curvature on the transformed surface only in the case of the right circular cylinder, and the resulting surface is any surface of revolution.

Mr. Whittemore's paper gives a short account of Hermite's investigations on quadric surfaces osculating a given surface. Hermite showed that, in general, at a finite number of simple points at any surface, the surface is osculated by all the members of a one parameter family of quadrics. In the present paper it is proved that a developable surface is osculated at every simple point by all the members of a one parameter family of quadrics. It is then shown that all the quadrics osculating a developable at any point are cones having a common vertex which is on the edge of regression of the developable. A complete geometrical definition of these cones is then obtained.

In a recent note Professor Lovett has shown that the various contact transformations of ordinary space which change straight lines into spheres are determined by pairs of bilinear equations between the point coördinates of the corresponding spaces, and that they fall into two categories according as all or certain determinants of a certain matrix reduce to constants. In the first of Professor Lovett's present papers the explicit forms of all the transformations of one category are given. There are thirty-one types, two of which depend on solutions of equations of the sixth degree; three on quintics; one on a pair of quartics; two on cubics; two are linear; and the remainder quadratic. To satisfy the equations of condition for the second category is in general impossible. However, at least two types of ∞^{15} transformations each are included in this category; they are generated by projective or dualistic transformations accompanied by Lie's transformation; they change asymptotic lines into lines of curvature.

Professor Lovett's second note solves four problems in the differential geometry of $n + 1$ dimensioned space. Of the infinite groups of contact transformations of this space, the following are determined: 1° those leaving the ratios of the Gaussian curvatures of any number of tangent surfaces invariant; 2° those making the curvature of corresponding surfaces at corresponding points equal; 3° those changing applicable surfaces into such; 4° those leaving invariant the equation

$$\Sigma_1^n \Sigma p_{ij} \, dx_i dx_j = 0.$$

In the third problem use is made of a remark of Staeckel, *Comptes rendus*, 1895; the first and second generalize results obtained by Mehmke and Vivanti, *Zeitschrift für Mathematik und Physik*, 1891, 1892; *Rendiconti del Circolo matematico di Palermo*, 1891.

Peano, in *Mathesis*, volume 9, page 75 and page 110, seems to have been the first to point out that the identical vanishing of the Wronskian of n functions of a single variable is not, in all cases, a sufficient condition for the linear dependence of these functions. (It is, of course, a necessary condition, provided the functions have finite derivatives of the first $n - 1$ orders at every point of the region in question.) At the same time he indicated a case in which it *is* a sufficient condition, and suggested the importance of finding other cases of the same sort. Without at first knowing of Peano's work, Professor Bôcher was recently led to the same question and found a case not included in Peano's, in which the identical vanishing of the Wronskian is a sufficient condition. In the present paper, which will appear in the *Transactions*, these cases and others of a similar character are discussed.

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ON SOME BIRATIONAL TRANSFORMATIONS OF THE KUMMER SURFACE INTO ITSELF.

BY DR. J. I. HUTCHINSON.

(Read before the American Mathematical Society, December 28, 1900.)

VERY few examples of the birational transformation of surfaces into themselves are as yet known in which the group is of infinite order.

The case of a continuous group with a finite number of parameters has been fully worked out,* but for discontinuous groups only two or three isolated examples † have up to the present time been met with.

It is in view of this, as well as on account of the general interest which attaches to the 16-nodal quartic, or Kummer, surfaces that I propose to show how to determine two groups of birational transformations of infinite order for which these surfaces are invariant.

In the first place I suppose the surface to be referred to a tetrahedron whose vertices are four nodes so chosen that none of the faces of the tetrahedron are singular tangent planes of the surface. Using homogeneous coördinates w, x, y, z, take for example the tetrahedron

$$(1) \qquad \begin{aligned} w &= a\vartheta_5^2 - b\vartheta_0^2, & y &= a\vartheta_{12}^2 - b\vartheta_{34}^2, \\ x &= b\vartheta_5^2 - a\vartheta_0^2, & z &= b\vartheta_{12}^2 - a\vartheta_{34}^2, \end{aligned}$$

where $\vartheta_\lambda = \vartheta_\lambda(u, v)$, and

$$a = c_{23}^2 = \vartheta_{23}^2(0, 0), \quad b = c_{14}^2.$$

The subscripts are here written according to the Weierstrass notation for the theta functions. The four functions ϑ_λ used in equations (1) form a Göpel quadruple and hence satify the well known Göpel biquadratic relation which I will indicate by

$$K(\vartheta_5, \vartheta_0, \vartheta_{12}, \vartheta_{34}) = 0.$$

The left member, regarded as a function $K(u, v)$ of u and v, vanishes identically. Solving (1) for ϑ_λ and substituting in this relation, we obtain the required equation of the

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* For an interesting account of this subject see Painlevé, Théorie analytique des équations différentielles, Paris, 1897.

† See Humbert, *Comptes rendus*, vol. 126, pp. 394, 508 ; and Painlevé, ibid., p. 512.

Kummer surface referred to the tetrahedron w, x, y, z. Writing for brevity

$$\alpha = c_{5}^{\frac{1}{2}}, \quad \beta = c_{34}^{\frac{1}{2}}, \quad \gamma = c_{12}^{\frac{1}{2}}, \quad \delta = c_{0}^{\frac{1}{2}};$$
$$l = \alpha\beta - \gamma\delta, \quad L = lmn(\alpha\beta + \gamma\delta),$$
$$m = \alpha\gamma - \beta\delta, \quad M = lmn(\alpha\gamma + \beta\delta).$$
$$n = \alpha\delta - \beta\gamma, \quad N = lmn(\alpha\delta + \beta\gamma),$$
$$P = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)\,(\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2)$$
$$- 2\alpha\beta\gamma\delta(\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + 4\alpha\beta\gamma\delta),$$

the required equation becomes

$$l^2m^2(w^2x^2 + y^2z^2) + n^2l^2(w^2y^2 + x^2z^2) + m^2n^2(w^2z^2 + x^2y^2)$$
$$- 2L(wx + yz)\,(wy + xz) + 2M(wx + yz)(wz + xy)$$
$$+ 2N(wy + xz)(wz + xy) + 2Pwxyz = 0.$$

It is at once apparent that this equation is reproduced when we perform the birational transformation

$$(B) \qquad w' : x' : y' : z' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Since there are 60 tetrahedra of the kind defined by equations (1), there are 60 transformations of the same type as (B). These generate a group G of infinite order. A much smaller number of operations, however, is sufficient to generate the same group, as we now proceed to show.

Consider the well known group G_{16} of linear transformations, for which the Kummer surface is invariant. These operations either leave the tetrahedron of reference T unchanged or permute it with three others. Denoting these by T_1, T_2, T_3, let t_1, t_2, t_3 represent linear transformations of G_{16} which permute T with T_1, T_2, T_3, respectively. Then $t_3 = t_1t_2$.

If B_i denotes the transformation of the same type as B associated with the tetrahedron T_i, then, since t_i is of period 2, it is evident that

$$B_i = t_i\,B\,t_i.$$

It follows from this that the 60 birational transformations of the type B can be generated by 15 of them properly chosen, together with the two linear transformations t_1 and t_2.

The group G can be enlarged by combining with it the transformations of G_{16} which leave the tetrahedron T unchanged (the faces, of course, being permuted). These transformations can be represented by

$$1,\ t',\ t'',\ t't'',$$

and the group G is of index 4 under the enlarged group.

It now remains to be shown that G is of infinite order. Consider the tetrahedron w_1, x_1, y_1, z_1, where

$$w_1 = \varkappa w + \lambda x, \quad y_1 = \rho w + \sigma x + \tau z,$$

$$x_1 = \lambda w + \varkappa x, \quad z_1 = \sigma w + \rho x + \tau y,$$

in which

$$\varkappa = c_{01}^2 c_{03}^2, \quad \lambda = c_2^2 c_4^2,$$

$$\rho = -c_2^2 c_{01}^2, \quad \sigma = -c_4^2 c_{03}^2, \quad \tau = c_{03}^4 - c_2^4.$$

The transformation *

$$(B_1) \qquad w_1' : x_1' : y_1' : z_1' : = \frac{1}{w_1} : \frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1},$$

written in terms of w, x, y, z, is

$$w' : x' : y' : z' = x : w : -\frac{\tau wx + y(\rho w + \sigma x)}{\sigma w + \rho x + \tau y}$$

$$: -\frac{\tau wx + z(\sigma w + \rho x)}{\rho w + \sigma x + \tau z}.$$

In a similar manner, by using the tetrahedron

$$w_2 = \mu w + \nu x, \quad y_2 = \xi w + \eta x + \zeta y,$$

$$x_2 = \nu w + \mu x, \quad z_2 = \eta w + \xi x + \zeta y,$$

where

$$\mu = c_4^2 c_{01}^2, \quad \nu = c_2^2 c_{03}^2,$$

$$\xi = -c_4^2 c_{03}^2, \quad \eta = c_2^2 c_{01}^2. \quad \zeta = c_{03}^4 + c_{01}^4,$$

we obtain the transformations

$$(B_2) \quad w' : x' : y' : z' = x : w : \frac{a(w^2 + x^2) + \beta wx + y(\gamma x - \delta w)}{\delta x - \gamma w + \epsilon y}$$

$$: \frac{a(w^2 + x^2) + \beta wx + z(\gamma w - \delta x)}{\delta w - \gamma x + \epsilon z},$$

where

$$a = 2c_2^2 c_1^2 c_{01}^2 c_{03}^2, \qquad \beta = (c_4^4 - c_2^4)(c_{01}^4 - c_{03}^4),$$

$$\gamma = c_4^2 c_{03}^2 (c_2^4 + c_4^4), \quad \delta = c_2^2 c_{01}^2 (c_2^4 + c_4^4),$$

$$\epsilon = (c_2^4 + c_4^4)^2.$$

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* The transformation B_1 here indicated is not the same as the B_1 previously referred to.

Combining these two transformations we obtain

$$(B_1B_2) \qquad w' : x' : y' : z' = w : x : \frac{Ay + B}{Cy + D} : \frac{A'z + B'}{C'z + D'},$$

where

$$A = -\varepsilon\tau wx + (\delta w - \gamma x)(\rho x + \sigma w),$$

$$B = -\,[\tau wx(\delta x - \gamma w) + \alpha(\rho x + \sigma w)(w^2 + x^2)$$
$$+ \beta wx(\rho x + \sigma w)],$$

$$C = \varepsilon(\sigma x + \rho w) + \tau(\gamma x - \delta w),$$

$$D = (\sigma x + \rho w)(\delta x - \gamma w) + \tau\alpha(w^2 + x^2) + \tau\beta wx.$$

The question of the periodicity of this transformation is clearly the same as that of the linear fractional transformation

$$y' = \frac{Ay + B}{Cy + D},$$

where A, B, C, D are independent of y or y'. But in order that this transformation may have a finite period n it is necessary and sufficient that*

$$(2) \qquad (A + D) = 4(AD - BC)\cos^2\frac{\lambda\pi}{n}$$

where λ is some integer relatively prime to n. It is easily seen that the expressions given above for A, B, C, D cannot satisfy such a relation as this. Hence the period of B_1B_2 is infinite.

If (u, v) are the hyperelliptic coördinates of a point of the Kummer surface, the equations

$$w = \theta_1(u, v), \quad x = \theta_2(u, v), \; y = \theta_3(u, v), \quad z = \theta_4(u, v),$$

where for brevity $\theta_i(u, v)$ are written for the right members in (1), express the homogeneous coördinates of a point of the surface in terms of the parameter coördinates of the same point. After the transformation B, the coördinates can be represented in the same form

$$w' = \theta_1(u', v'), \; x' = \theta_2(u', v'), \; y' = \theta_3(u', v'), \; z' = \theta_4(u', v'),$$

where (u', v') are the parameters of the point into which the point (u, v) has been transformed.

A question arises as to what are the relations between u',

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[*] See Serret, Cours d'algèbre supérieure.

v' and u, v. These cannot be algebraic, for if they were, say

$$A_1(u', v', u, v) = 0, \quad A_2(u', v', u, v) = 0,$$

then since to the point $(1, 0, 0, 0)$ corresponds the curve of intersection of $w' = 0$ with the Kummer surface, when we substitute for u, v in these relations the parameter values for this point the two equations ought to be equivalent to a single algebraic relation

$$A(u', v') = 0.$$

This latter should then be the equation of the curve which corresponds to the given point. But the equation of this curve is

$$\theta_1(u', v') = 0,$$

a relation which is not algebraic but transcendental. In fact, the relations between u', v' and u, v are none other than those which at once follow from the equations of transformation[*] (B), viz.:

$$(3) \quad \theta_i(u', v')\theta_i(u, v) = \theta_j(u', v')\theta_j(u, v), \quad [i, j = 1, 2, 3, 4].$$

It would be interesting to know whether or not the group G is discontinuous. It appears very likely that it is, although I have not been able to fully settle this question. It can be shown that, if G contains any infinitesimal transformations, they belong to an invariant subgroup of G, and that they leave unchanged each point of the Kummer surface. For any combination of the generating operations B_i can, according to (3). be represented in the form

$$\frac{\theta_i(u, v)}{\theta_j(u, v)} = \frac{\Phi_i(u', v')}{\Phi_j(u', v')},$$

where $\Phi_i(u', v')$ is a theta function of order $2n$, whose form it is not necessary to determine. If now

$$u' = u + \xi, \quad v' = v + \eta,$$

where ξ and η are infinitesimals, we have, on expanding,

$$\frac{\theta_i(u, v)}{\theta_j(u, v)} = F_{ij}(u, v) + \frac{\partial F_{ij}}{\partial u}\xi + \frac{\partial F_{ij}}{\partial v}\eta + \cdots,$$

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[*] It is from a similar point of view that Humbert has remarked the existence of birational transformations of the Kummer surface into itself. See *Liouville's Jour.*, 1893, p. 466.

where

$$F_\nu = \frac{\phi_i}{\Phi_j}.$$

From this follows that

$$F_\nu(u, v) = \frac{\theta_i(u, v)}{\theta_j(u, v)},$$

and

$$\frac{\partial F_\nu}{\partial u}\xi + \frac{\partial F_\nu}{\partial v}\eta = 0.$$

From the latter condition follows that the Jacobian of any two of the functions $\frac{\theta_i(u, v)}{\theta_j(u, v)}$ vanishes and hence that all are functions of one among them. But this is clearly impossible; hence if an infinitesimal transformation occurs in G it leaves every point of the Kummer surface unchanged. The totality of operations in G for which every point of the surface is invariant evidently form a self-conjugate subgroup of G. Moreover it is clear that this subgroup cannot be a finite continuous group since the Kummer surface does not enter into the category of surfaces which admit such groups.

Another group G' of birational transformations of the Kummer surface into itself is determined from the fact that a one to one correspondence exists between this surface and the Weddle surface (locus of the vertex of a quadric cone which passes through six fixed points). Hence a birational transformation of the one corresponds to the like of the other. If we write

$$w : x : y : z = \vartheta_{01}\vartheta_{12}\vartheta_5 : \vartheta_{23}\vartheta_{03}\vartheta_6 : \vartheta_{23}\vartheta_{12}\vartheta_4 : \vartheta_{03}\vartheta_{01}\vartheta_4$$

$$a : b : c : d = c_{01}c_{12}c_5 : c_{23}c_{03}c_5 : c_{23}c_{12}c_4 : c_{03}c_{01}c_4,$$

$$\alpha : \beta : \gamma : \delta = c_{23}c_{03}c_4 : c_{01}c_{12}c_4 : c_{01}c_{03}c_5 : c_{12}c_{23}c_6,$$

the equation of the Weddle surface is

$$\begin{vmatrix} xyz & w & a & \alpha \\ wyz & x & b & \beta \\ wxz & y & c & \gamma \\ wxy & z & d & \delta \end{vmatrix} = 0.$$

This equation is unchanged for a transformation of the same form as (B). I will denote this transformation by C. Since the equation of the surface can be written in 15 different ways in this form, we have 15 corresponding transformations which generate a group of infinite order. For consider the tetrahedron w_1, x_1, y_1, z_1, where

$$w_1 = w, \quad (b + a)x_1 = bw - ax,$$
$$(c + a)y_1 = cw - ay, \quad (d + a)z_1 = dw - az.$$

The associated transformation is

$$w_1' : x_1' : y_1' : z_1' = \frac{1}{w_1} : \frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1},$$

or

$$w' : x' : y' : z' = \frac{1}{w} : \frac{bx + (a + 2b)w}{awx - bw^2} : \frac{cy + (a + 2c)w}{awy - cw^2}$$
$$: \frac{dz + (a + 2d)w}{awz - dw^2}.$$

Denoting this transformation by C_1 we have for $C_1 C$

$$w' : x' : y' : z' = w : \frac{(a + 2b)wx + bw^2}{aw - bx} : \cdot$$

Here x' is proportional to an expression of the form

$$\frac{Ax + B}{Cx + D}.$$

It is easily seen that

$$\frac{(A + D)^2}{AD - BC} = 4,$$

and hence the condition (2) reduces to

$$\cos^2 \frac{\lambda\pi}{n} = 1.$$

But since $0 < \lambda < n$ this relation cannot be satisfied for a finite value of n. Hence the transformation $C_1 C$ has an infinite period.

CORNELL UNIVERSITY,
December, 1900

THEOREMS CONCERNING POSITIVE DEFINITIONS OF FINITE ASSEMBLAGE AND INFINITE ASSEMBLAGE.

BY MR. C. J. KEYSER.

(Read before the American Mathematical Society, December 28, 1900.)

THE well known wide disagreement* among workers in assemblage theory as to the meaning that should be attached to certain indispensable terms, such as assemblage, defined assemblage, given assemblage, law of formation, and so on, indicates alike the need and the possibility of a more critical grounding of this doctrine. Here as elsewhere the guiding principle of criticism should be the principle of Parsimony, the so-called Razor of Occam : Entia non sunt multiplicanda præter necessitatem. For the problem is one of simplification, of logical economy, of minimizing hypothesis. The ideal is, in a word, to dispense with the undefined. Now nothing seems clearer than that no amount of criticism, however acute, can completely eliminate the undefined from the foundations of knowledge. For every explicit involves some implicit, and so there must be not merely assumption, but implicit and therefore undefined assumption. Whence appears that a residuum of indetermination and doubt must elude even the most refined and searching analysis, and ultimate simplicity, perfect certainty, absolute rigor, remain beautiful dreams, destined never to be quite realized. Nevertheless the problem admits of closer and closer approximate solution, which, theoretically at least, takes the form of an unending series, namely, of substitutions of simpler (there is no simplest) undefined for the less simple. Unfortunately the process appears to conduct sooner or later from seeming light and certainty into the "frightful shadow-land of metaphysics." But it is only appearance and only seeming, for it is the common lot, consciously or unconsciously, to dwell in that land always Shadows critical reflection undoubtedly produces, but shadows are incident to the illumination of darkness. Cousin is right : La critique est la vie de la science.

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* Cf. J. Tannery : "De l'infini mathématique," *Revue générale des sciences*, vol. 8 (1897). Dedekind : Was Sind und Was Sollen die Zahlen, § 3, Art. 32 ; § 5, Art. 64 G. Cantor : "Sur les ensembles infinis et linéaires de points," *Acta Math.*, vol. 2. Couturat : De l'infini mathématique, Appendice, Note IV. E. Borel : Leçons sur la théorie des fonctions, Chap. I. J. Tannery : Introduction à la théorie des fonctions d'une variable, § 15.

This paper presents some results of an attempt, consistently with the mentioned principle of criticism, to show that definitions, of a more general character than those usually considered to be mathematically available, are in fact sufficient for the establishment of certain fundamental propositions of assemblage theory.

I. *Hypotheses, and Definition of Terms.*

It is assumed that, if there be anything which throughout some definite time is intrinsically regardable as an individual, as one, this intrinsic capability persists throughout every definite time. Any such thing will be named *element*. It is further assumed : (1) that there are elements ; (2) that one of the elements is the idea ϵ, that each of the elements, including ϵ, *is* an element ; (3) that, if two elements be distinguishable, the mark of distinction is an element.

The element ϵ, serving as a bond of union of all elements, strictly defining all as an individual, as the sole invariant content of the element category, imparts sufficient precision to the expression totality of elements. This totality, which might be called the universe of elements, may be denoted by U. U being, by definition, itself an element, is to be regarded as contained in itself.

Two elements are *identical* if they are indistinguishable from each other. Two such elements are, for all logical purposes, for attention, not two but one ; indeed, they can not even be counted as two, for to suppose them so countable is to suppose them capable of arrangement in the time-order of before and after, which is tantamount to supposing them distinguishable. Hence U contains no identical elements : *there are no identical elements.*

An element that is two or more elements regardable as an individual may be called a *composite* element ; in the contrary case, an *ultimate* or *simple* element.

Now let e and e' be any two elements. There is at least one mark of distinction. Represent it by the prime $'$. This mark is, by (3), itself an element. Consequently e' is a composite element. Similarly for e. Like reasoning applied to the element $'$ will show it to be composite, and so on indefinitely ; indefinitely, for if the process exemplified could terminate, it would end with an element having no mark of distinction, *i. e.*, with a mark of distinction itself without distinction. Such a mark would be an element identical with some (in fact every) other. Hence all elements are composite : *there are no simple or ultimate elements.*

It thus appears that an element is a shield with two sides, one fact F with two phases: the phase of being constituted of elements, constituents; and the phase of being a bond uniting the latter into one, an individual. The two phases of F may be briefly called respectively the *multi*-phase and the *uni*-phase. Neither is logically prior. Either may be regarded as *primary*, the other as *secondary*—the choice to be determined by considerations of expedience. F will be called element or assemblage (of elements) according as the uni-phase or the multi-phase of F be taken as primary. The notions, element and assemblage, are, therefore, *reciprocal** and complemental, each is implicit in, incomplete without, determined by, the other.

This principle of reciprocity enables us to say: all elements and assemblages are *defined*. If F be an assemblage, its definition is its uni-phase, its element-character; if F be thought as an element, its definition is its multi-phase, its character as a certain composite. For easier comparison of this with the corresponding conceptions of Cantor and Dedekind,† we state it at length thus: Every assemblage is defined in the sense that the principle of excluded middle validates, with respect to any assemblage, the proposition: any element whatever is a constituent of the assemblage or it is not. Reciprocally, every element is defined in the sense that the same principle justifies, in regard to any element, the proposition: the constituents of any assemblage whatever compose the element or they do not.

The term "defined" is here used in its *objective*, which is at the same time its fundamental and etymological, sense of bounded or marked off from. Every such objective definition is an F with its two phases. Its essence is being, not actually regard*ed* but merely regard*able*. When actually regarded, made the recipient of attention, then, lying in consciousness or expressed in language or both, it becomes an ordinary, *subjective*, definition. When so expressed the corresponding F (assemblage or element) may be said to be *given*. Discussion of the exceeding difficult question‡ of "given" is reserved for a future communication as is also a fuller exhibition of the cardinal importance of the principle of reciprocity. In regard to the latter we merely add here the now rather obvious facts: that assumptions (1), (2), and (3) involve like assumptions respecting assemblages; that there are no identical assemblages; that, as every assemblage has its uni- and every element its multi-phase, there is one and but one element for each assemblage and reciprocally; and, that, in particular, U plays the double rôle of being at once the totality of elements and the totality of assemblages, including itself in both characters.

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* It is noteworthy that the so-called "principle of reciprocity" which manifests itself familiarly in various specialized and concrete ways, notably in geometry, is in all strictness a *principle* and is such for assemblage theory in general, being inherent, as just shown, in the inmost core of the discipline

† Cf. Cantor and Dedekind: *Op. cit.*

‡ In relation to this matter, the first of J. Tannery's essays already cited may be consulted with much profit.

If E and E' are any two assemblages such that there is an assemblage M having the properties: (1) each element of M is a pair of elements e and e', where e and e' signify element of E and element of E' respectively; (2) each element e occurs once and but once, and no element e' occurs twice, in M; the assemblage E may be said to be *projectible* upon the assemblage E'. As in M each element e of E is associated with a certain element e' of E', the assemblage M may be said to indicate a mode or *manner* of the projectibility; *a* rather than *the* manner, because obviously, if E be projectible upon E' in one manner, it will in general be projectible upon E' in more than one manner. If, in a given M, e_k' of E' is associated with e_k of E, e_k' may be called the *projection* of e_k for the given M. The totality of such projections is an assemblage which may be denoted by P.

If two assemblages E and E' are each projectible upon the other in a manner M, E and E' may be said to be *congruent* in the manner M, or a unique and reciprocal correspondence may be said to exist in the manner M between E and E'. If two assemblages E and E' be congruent, the relationship may be stated symbolically by writing

$$E \equiv E'.$$

If every element of an assemblage A is an element of an assemblage A' and if one or more elements of A' are not in A, A will be called a *part* (partie intégrante of Cantor, echter Theil of Dedekind) of A', and we may write

$$A < A'.$$

If this relation subsists between A and A', then plainly A is projectible upon A'. Also any E is congruent with itself. The equation

$$A_1 + A_2 = A$$

may be employed to indicate at once that A_1 and A_2 are each a part of A, that A_1 and A_2 have no element in common, and that every element of A is an element of A_1 or of A_2.

II. *Demonstration of Theorems.*

Let E be an assemblage. Either E is congruent with an assemblage

$$(N) \qquad 1, 2, \quad, n,$$

where n is some definite integer, or such is not the case. If E be not congruent with (N), the assemblage

$$(N') \qquad 1, 2, \cdots, n, \cdots$$

is projectible upon E. To prove this it is sufficient to prove that, however large a value n' of n be chosen in advance, the assemblage

$$1, 2, \cdots, n', n' + 1$$

is projectible upon E. That this assemblage is projectible upon E follows immediately from the fact that, were it not so projectible, E would itself be projectible upon (N) for $n \leqq n'$, which is contrary to hypothesis. Let the assemblage

$$(M) \qquad (1, e_1), (2, e_2), \cdots, (n, e_n), \cdots$$

indicate a manner of projectibility of (N') upon E. Then the assemblage P of projections will be

$$e_1, e_2, \cdots, e_n, \cdots$$

and we may write

$$E = P + R$$

where R denotes the assemblage of those elements of E which, if there be any such, are not elements of P. Represent the assemblage

$$e_2, e_4, \cdots, e_{2n}, \cdots$$

by P_1. It is plain that P is projectible upon P_1 in the manner

$$(M_1) \qquad (e_1, e_2), (e_2, e_4), \cdots, (e_n, e_{2n}), \cdots$$

and R is projectible upon itself; hence the assemblage $P + R$ is projectible upon the assemblage $P_1 + R$; but

$$P + R = E \quad \text{and} \quad P_1 < P$$

whence

$$P_1 + R < E.$$

E is therefore projectible upon a part of itself. As any assemblage is obviously congruent with the assemblage of its projections, E is seen to be congruent with a part of itself. Accordingly we may state the

THEOREM (*a*). *If any assemblage E be such that there is no definite value of n for which the assemblage*

$$1,\ 2,\ \cdots,\ n$$

is congruent with E, then E is congruent with a part of itself.

From this follows directly the contrapositive

THEOREM (*b*). *If any assemblage E be not congruent with a part of itself, then, for some definite value of n, the assemblage*

$$1,\ 2,\ \cdots,\ n$$

is congruent with E.

Are the converses of these propositions correct? Suppose that

$$A < E, \text{ and that } E \equiv A \text{ in some manner } M.$$

It is required to determine whether there is a determinate integer n such that the assemblage

$$(N) \qquad 1,\ 2,\ \cdots,\ n$$

is congruent with E.

Let e_k denote any definite element of E. In M this element is paired with some definite element e_j of E, whence e_j is an element of A. Now, if E is congruent with (N) in one or more manners, let M' stand for any definite one of them. In M' the element e_k has for its projection, or is paired with, some definite integer n' of (N) where of course it is undetermined whether n' is less than or equal to n. Accordingly, M and M' together furnish a scheme for associating in a one to one fashion the elements of A with those of (N): given any element, as e_k of E, those two elements, e_j and n', of A and (N), respectively, shall be associated, which are associated with e_k in M and M', respectively. Denote by M'' the assemblage of such pairs e_j, n'. It thus appears that if

$$E \equiv A$$

in some definite manner M', then

$$A \equiv (N)$$

in a definite related manner M''. But since

$$A < E,$$

there is at least one element e_i of E which is not an element of A, and, therefore, no one of such elements e_i enters a pair in M''. Hence A is not congruent with (N) in the manner M''; whence follows that E is not congruent with A in the manner M', and, therefore, not in any manner.

We have, accordingly, the following propositions:

THEOREM (a'). *If any assemblage E is congruent with a part of itself, there is no assemblage*

$$1, 2, \cdots, n \qquad (n \textit{ definite})$$

with which E is congruent.

THEOREM (b'). *If there is an assemblage*

$$1, 2, \cdots, n \qquad (n \textit{ definite})$$

with which an assemblage E is congruent, E is not congruent with a part of itself.

These reciprocally contrapositive propositions are the converses, respectively, of (a) and (b).

III. *Logical Significance of the Four Theorems.*

Denote by U the assemblage of all assemblages, including U. The elements of U are the component assemblages of U. Of these there are two, F_1 and I_1, such that the former signifies the assemblage of all assemblages which are not, and the latter signifies the assemblage of all assemblages which are congruent each with a part of itself. By the principle of excluded middle,

$$U = F_1 + I_1.$$

By the same principle,

$$U = F' + I',$$

where F' denotes the assemblage of all assemblages of which each has, and I' denotes the assemblage of all assemblages of which none has, the property of being projectible upon (or congruent with) an assemblage

$$1, 2, \cdots, n \qquad (n \text{ definite}).$$

Now the logical import of the propositions (a), (b), (a'), and (b') is that F_1 and F', as also I_1 and I', are identical, element for element. Accordingly, we may define as follows:

(1) *A finite assemblage is one which is not congruent with any part of itself.*

(2) *An infinite assemblage is one which is not congruent with any assemblage.*

$$1, 2, \cdots, n \qquad (n \textit{ definite}).$$

Or quite equivalently thus:

(3) *A finite assemblage is one which is congruent with some assemblage*

$$1, 2, \cdot\ , n \qquad\qquad (n \textit{ definite}).$$

(4) *An infinite assemblage is one which is congruent with a part of itself.*

In neither pair of definitions are the notions finite and infinite *formally* represented as mutually exclusive or as negatives of each other; while the second pair has the additional advantage of representing both the notions in question as *positive** concepts.

Before passing to the final section of this paper, it seems worth while to pause long enough to glance, from the present point of view, at the notion, cardinal number of a finite assemblage, and to offer a proof that this notion, for any given finite assemblage, possesses the property of invariance.

Consider the *ordered* assemblage

$$(S) \qquad\qquad 1, 2, \cdots, n', \cdots, n, \cdots n'',$$

where n', n and n'' are definite and subject to the condition that, in (S), n' comes before n and the latter before n''. Denote by $S_{n'}$, S_n, and $S_{n''}$ the assemblages obtained by supposing in succession that (S) ends with n', n and n'', and in each case abstracting from the order of the elements. $S_{n'}$, S_n, and $S_{n''}$, being congruent each with itself, are finite assemblages, and, as by definition

$$S_{n'} < S_n \quad \text{and} \quad S_n < S_{n''},$$

we have by (3)

$$\text{neither } S_{n'} \equiv S_n \text{ nor } S_n \equiv S_{n''}.$$

Now let E be any finite assemblage. By (3) there is some S_n for which

$$E \equiv S_n.$$

We now *define* n to be the cardinal number of the assemblage E. Plainly, this number is independent of every arrangement of the elements of E; for if, by any particular arrangement,

$$E \equiv S_{n'} \text{ or } E \equiv S_{n''},$$

we would have also

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* As to this phase of the matter and also in relation to definitions (1) and (4), cf. Jules Tannery, *op. cit.*; Dedekind, *op. cit*; G. Cantor, *op. cit.*; Couturat, *op. cit.*

$$S_n \equiv S_{n'} \text{ or } S_{n''} \equiv S_n .$$

It readily follows that any two congruent finite assemblages have the same cardinal number.

IV. *Connection with the Logic of Relatives: Another Point of View.*

Consider the argument:

Every element of an assemblage E is projected upon some other element of E;

No element of E is the projection of more than one element of E;

.·. Every element of E is a projection of an element of E; *i. e.*, the assemblage of projections is congruent with E.

This argument is of the type of the so-called syllogism of transposed quantity, a mode of ratiocination of which the earliest scientific account is found in the Logic of De Morgan. Its uses and limitations have been pointed out by Mr. C. S. Peirce,* to whom is due also its perfection as an instrument of research as well as the indication of its significance in modern logic as such. From definition (1) or from the last proposition of section III, it is seen that the conclusion is warranted by the premises always when and only when E is a finite assemblage. Reversing the order of ideas, the above argument, regarded as purely a *form*, may be taken as the 'logical' discriminant of the assemblage

$$U = F_1 + I_1$$

and accordingly we might define: any given assemblage is finite or infinite, belongs to F_1 or to I_1, according as the syllogism is valid or not valid for that assemblage. Such appears to be in fact Mr. Peirce's point of view in one or two instances. At all events, it requires merely verbal substitution to render the syllogism in question quite identical with the "Hottentot" and "Texan" arguments employed by him in the works cited below. The interest here attaches, of course, to the demonstrated ultimate equivalence of the two points of view.

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*Peirce: "On the Algebra of Logic," *Amer. Jour. of Math.*, vol. 7, p. 201 "On the Logic of Number," *Amer. Jour. of Math.*, vol 4, p. 75. Definition of "finite," Century Dictionary.

DINI'S METHOD OF SHOWING THE CONVERGENCE OF FOURIER'S SERIES AND OF OTHER ALLIED DEVELOPMENTS.

BY MR. WALTER B. FORD.

(Read before the American Mathematical Society, December 28, 1900.)

If it is required to show that the arbitrary function $f(x)$ of the real variable x may be developed into a Fourier series for all values of x lying between $-\pi$ and π ($-\pi$ and π at most excluded) then, remembering that for any particular value of x, such as $x = a$, the sum of the first $n + 1$ terms of the series is

$$(1)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}\,dx,$$

it is easily shown that we need merely examine the limits approached by the integrals

$$(2)\qquad \int_{0}^{d} f(x)\frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}\,dx,$$

$$(3)\qquad \int_{c}^{d} f(x)\frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}\,dx,$$

as n increases indefinitely, c and d being any numbers such that $0 < c < d \gtreqless \frac{1}{2}\pi$. In case these limits exist and have certain simple properties it becomes evident that the given series will be convergent for any value $x = a$ which lies between $-\pi$ and π, and will have as its sum either $f(a)$ or

$$\frac{f(a+0)+f(a-0)}{2},$$

while at either of the points $x = -\pi$ or $x = \pi$ the sum will be

$$\frac{f(-\pi+0)+f(\pi-0)}{2}$$

Moreover, a closer examination of integrals (2) and (3) shows that if $f(x)$ satisfies any one of certain sets of preliminary conditions (of which Dirichlet's conditions are a special case), then these integrals (2) and (3) possess the required properties; for in such cases the problem reduces to the establishing of certain well known properties of the more simple integrals

$$(4) \qquad \int_0^d \frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}\,dx,$$

$$(5) \qquad \int_c^d \frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}\,dx,$$

where c and d have the same significance as before. In such cases, then, the original problem reduces to questions which are entirely independent of the function $f(x)$ which was to be developed. Now, integrals (4) and (5) being once closely examined and their properties which are essential to the present problem being once tabulated, the question arises whether there are not still other functions of x, n and a, which like

$$\frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)},$$

when integrated between certain determined or undetermined limits $(0, d')$ in the one case and (c', d') in the other, will possess the same essential properties which belong to (4) and (5), and, as thus integrated, will therefore form integrals bearing the same relation to *new* series developments of $f(x)$ for x between certain limits (a, b) that (4) and (5) bore to the Fourier development of $f(x)$ for x between $-\pi$ and π. Evidently if such functions exist, representing any one of them by $\varphi(x-a, a, n)$, or better by $\varphi(x-a, a, h_n)$ where h_n is any particular expression which (like the $\frac{2n+1}{2}$ which appears in (4) and (5)) is always positive and increases indefinitely with n, the integrals in question will be

$$(6) \qquad \int_0^{d'} \varphi(x-a, a, h_n)\,dx,$$

$$(7) \qquad \int_{c'}^{d'} \varphi(x-a, a, h_n)\,dx,$$

and for any particular value of x such as $x = a$ it is evident that the sum of the first $n + 1$ terms of this new series will be an integral corresponding to the integral (1), and in fact it may be easily shown that this sum will be

$$(8) \qquad \frac{1}{2G}\int_a^b f(x)\varphi(x - a, a, h_n)dx,$$

where G is a determinate constant independent of x and of a and different from zero.

Dini begins by showing that there are an infinite number of such functions $\varphi(x - a, a, h_n)$*. Then considering the series

$$(9) \qquad \frac{1}{2G}\int_a^b f(x)\varphi(x - a, a, h_0)dx$$

$$+ \frac{1}{2G}\sum_1^\infty \int_a^b f(x)\{\varphi(x - a, a, h_n) - \varphi(x - a, a, h_{n-1})\}dx,$$

the sum of whose first $n + 1$ terms is evidently the above expression (8), he proceeds to make farther assumptions as regards the functions $\varphi(x - a, a, h_0)$, $\varphi(x - a, a, h_1)$, $\cdots$ $\varphi(x - a, a, h_n)$; *i. e.*, he assumes that each of these functions is such that every term of (9) may be expressed in a certain typical form (which is the form common to all known developments). One of the chief requirements of a term in this form is that it shall involve in a specified way certain special functions of x, or of x and λ_n, where λ_n is a parameter variable only with n, which functions we may suppose to have taken in advance as the functions in terms of which any new development might be desired (these functions in the case of the Fourier series being sin nx and cos nx.)

This being done, it is evident that if $f(x)$ satisfies any one of the sets of conditions mentioned before, we may now form a sufficient condition that this function may be developed for all values of x lying between the determined or undetermined limits (a, b) (a and b at most excluded) in terms of given functions of x, or of x and λ_n, the development being of the form common to all known developments (developments such as occur, for example, in the general study of mathematical physics).

This condition is described by saying that the sum of the first $n + 1$ terms of the given series should be equal to an integral of the form (8), where $\varphi(x-a, a, h_n)$ is a function of the type mentioned above. But to use this condition as a criterion in any case where we desire to test a given development is evidently difficult, for we are thus required to determine certain facts about the nature of an infinite series whose individual terms are generally of a complicated character. As an aid to this undertaking, Dini now employs an ingenious method due in substance to Cauchy. He virtually shows how a function of the complex variable z may be constructed so that it will have an infinite number of poles of the first order distributed at finite distances from each other along the positive side of the axis of reals, this function being otherwise monogenic within finite regions of the z plane, and being, moreover, so constructed that its residua in these poles (or at least in some of them) are equal respectively to the individual terms of the given series in x which we are to test. This once accomplished, it is evident that we have merely to integrate this function of z about any closed contour lying to the right of the axis of pure imaginaries and enclosing the first $n+1$ of these poles and not passing through any one of them, in order to obtain readily in the form of a definite integral (in the complex) the sum of the first $n+1$ terms of the given series; and by enlarging the contour so as to enclose more and more of the poles, we shall have in the limit as the contour thus enlarges indefinitely an expression in the form of a definite integral (in the complex) which actually represents the sum of the given series. Thus, by this means our investigation is transferred from questions regarding the given series to questions regarding a definite integral which varies with n, and for any special value of n is equal to the sum of the first $n+1$ terms of the given series. Whenever this integral reduces to one of the integrals (8) we are assured that the given development is possible. To study this reduction is, however, difficult in many cases, but Dini shows how conclusive results may be obtained in the case of Fourier series, series in terms of zonal harmonics, Bessel functions, and elliptic functions, and he makes it evident that the method leads to decisive results in many other cases.

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* Reference is here made to Dini's work entitled: Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale; Pisa, 1880.

ANN ARBOR,
December, 1900.

SHORTER NOTICES.

Application de la Méthode Vectorielle de Grassmann à la Géométrie Infinitésimale. Par HENRI FEHR.* Georges Carré et C. Naud, Éditeurs. Paris, 1899. 94 pp.

THE mathematicians Grassmann and Hamilton who almost simultaneously published their first work upon a "calculus of space" were undoubtedly geometers, not analysts—abstract and speculative they may have been—but still geometers. We have only to regret that they were not also clear stylists; for it must be admitted that, with the exception of the attempts of certain persons to show how everything may be done by the methods of vector analysis or quaternions, nothing could be more fatal to the popular acceptation and use of a space analysis than the form in which it was presented to the public by the inventors. Succeeding writers for the most part seem to have erred along the same lines or to have forgotten the stress originally laid upon the interpretation of the analysis. It is therefore with great pleasure that we read M. Fehr's little book, which is written with such admirable clearness and selected with such tasteful care that in the compass of ninety-one pages there is included, without the slightest suspicion of crowding, a preface, an introduction on the use of vector analysis, and a fairly complete treatment of differential geometry.

M. Fehr originally wrote his book as a thesis to be presented for the degree of doctor of science at the University of Geneva. As a thesis, the work contains nothing original either in vector analysis or in geometry. All the results and methods were known well enough before. Indeed, anyone who has heard such lectures as are given at our leading universities upon these two branches of mathematics ought to be able to put together the material in this book with almost no difficulty. But to put it together in so pleasing a manner is a far harder task. The reviewer, therefore, must deal with the manner of presentation rather than with that which is presented.

The votaries of vector analysis will find in M. Fehr's book little to cheer them. In fact, so meager is the development of the subject that one might almost say the entire analysis consists of three symbols

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* Now professor of geometry and algebra at the University of Geneva.

$[a_1\ a_2] = Va_1 a_2$, the vector product of two vectors,
$[a_1 \mid a_2] = -Sa_1 a_2$, the scalar product of two vectors,
$[a_1\ a_2\ a_3] = -Sa_1 a_2 a_3$, the scalar product of three vectors.

These expressions are used, to all intents and purposes, as abridged notations. If two vectors a_1 and a_2 are perpendicular, the fact is indicated by writing $[a_1 \mid a_2] = 0$; if they are parallel, by $[a_1\ a_2] = 0$; and if three vectors a_1, a_2, a_3 are coplanar, by $[a_1\ a_2,\ a_3] = 0$. The working rules of the analysis and the formal transformations of the symbols are developed scarcely at all. M. Fehr knows very well what to omit, and he omits a great deal. The differentiating operator ∇, which can be used with considerable advantage in some portions of differential geometry, is not introduced at all. This omission tends to clearness and simplicity. The linear vector function is also left out to advantage. Indeed, it seems as if the formal methods of vector analysis are none too well suited to the development of differential geometry. The vectors appear more or less artificial. They are not thrust upon us irresistibly, as in physics. The need of a vector analysis is not so apparent as it might be. M. Fehr, therefore, has done well to introduce as little as possible of the vector machinery. Perhaps it is well to note that M. Appell has followed a similar course in his treatment of the theory of vectors: Traité de mécanique rationelle, volume I, chapter I.

The presentation of the elements of differential geometry given by M. Fehr is hardly to be excelled. The usual important and fundamental results are obtained in much less space and time than is customary, and yet with perhaps a gain instead of a loss in clearness. The reason for this unusual brevity is not so much the compactness of the vector notation as the concreteness of the vector idea. No time is wasted in developing analytical formulæ. The mind is brought to bear directly upon the geometric questions at hand, and they are solved. It is this constant appeal to visualization that shortens the work; it is this that distinguishes the book from others; it is this that adds so much of perspicuity; and it is this that leads us to recommend the book most heartily to all who teach or study this subject. The vector ideas will clarify and render definite the conceptions of geometry to an extent scarcely possible with other methods. The insignificant amount of vector analysis need cause no fear of difficulty.

Mr. Fehr is to be thanked for his excellent book, and Messrs. Carré et Naud for the clear and almost errorless typography. The following reproduction of the table of contents will serve to show more satisfactorily than anything else the scope of the work: *Préface, Introduction:* Rappel de quelques notions de calcul géométrique; *Des courbes gauches:* 1° Généralités, 2° Courbure et rayon de courbure, 3° Torsion, formules de Frenet, 4° Courbure normale, formule de Lancret; *De la théorie des surfaces:* 1° Génér-alitiés, 2° Relations fondamentales; *De la courbure des courbes tracées sur une surface:* 1° Théorème de Meusnier, 2° Courbure des sections principales, 3° Formule d' Euler; *De la courbure des surfaces:* 1° Courbure totale, application aux surfaces réglées, 2° Courbure moyenne; cas particulier, 3° Courbure moyenne quadratique; *Des lignes tracées sur une surface;* 1° Systèmes conjugués, 2° Lignes de courbure; théorème de Dupin, 3° Lignes asymptotiques, 4° Lignes géodésiques; courbure géodésique d'une ligne tracée sur une surface.

E. B. WILSON.

Annuaire pour l'Ar 1901, publié par le Bureau des Longitudes. Paris, Gauthier-Villars.

THE new *Annuaire* which has just been received contains this time but few alterations. In the astronomical portion, however, the article on the various kinds of calendars has been rewritten and enlarged, the rules for obtaining the date of Easter in any year being much improved in statement and completeness. The historical part containing the tables of small planets, comets, etc., is, of course, brought up to date. A useful addition is a curve showing graphically the amount of the equation of time for any period of the year; this has been carefully drawn by M. Cornu. In the scientific notices, M. Cornu resumes his article of last year under the title 'Le transport électrique de la force'; M. Poincaré gives the report of the committee appointed to consider the question of the revision of the Quito meridian arc; M. Bassot writes the history of the introduction and establishment of the metric system. MM. Loewy and Bouquet de la Grye respectively contribute accounts of the international astronomical and geodetic congresses held in Paris during the summer; the aeronautical congress is represented by M. J. Janssen's presidential address. The last named also gives his annual report of the work done in the Mont Blanc Observatory during the past year. The final notice contains the oration of M. le commandant Guyou delivered at the meeting held to commemorate the late M. de Bernardières.

The increase in size of the *Annuaire* is due to the larger number of scientific notices and to the insertion of nearly 100 pages of advertisements. The annual proper is only eight pages longer than its predecessor.

ERNEST W. BROWN.

ERRATA.

On page 121 of the present volume of the BULLETIN, in the last line of my paper, the word *column* should be changed to *row*. The slip of my pen which produced this error is the more to be regretted as it makes Peano's theorem appear almost identical with the theorem which I had stated above, whereas the two theorems are wholly distinct from each other.

MAXIME BÔCHER.

Other errata in the present volume which have come to the attention of the editors are:—

Page 11, line 3 and line 30, *for* conformably *read* conformally.

Page 136, line 27, *for* p *read* p_y.
Page 142, line 22, *for* r_1 *read* r_1^2.
Page 158, line 15, *for* pure by *read* purely.

NOTES.

A NEW Annual Register of the AMERICAN MATHEMATICAL SOCIETY has just been published and distributed among the members. Copies of the Register may be obtained from the Secretary.

THE first (January) number of Volume II of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Invariants of systems of linear differential equations," by E. J. WILCZYNSKI; "Divergent and conditionally convergent series whose product is absolutely convergent," by FLORIAN CAJORI; "Sets of coincidence points on the non-singular cubics of a syzygetic sheaf," by M. B. PORTER: "Note on non-quaternion number systems," by W. M. STRONG; "On the reduction of the general abelian integral," by J. C. FIELDS; "Ueber Flächen von constanter Gauss'scher Krummung," by DAVID HILBERT; "Note on the functions of the form $f(x) \equiv \varphi(x) + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ which in a given interval differ the least possible from zero," by H. F. Blichfeldt.

The January number of the *Annals of Mathematics* contains the following papers: "Extension of Hurwitz's proof for the transcendence of *e* to the transcendence of π," by R. E. Moritz; "An application of elliptic functions to Peaucellier's link-work (inversor)," by A. Emch; "Note on the geometrical treatment of conics," by C. A. Scott; "On two-dimensional fluid motion through spouts composed of two plane walls," by R. A. Harris; "On a special class of abelian groups," by G. A. Miller; "The theory of linear dependence," by M. Bôcher; "Brilliant points of a family of concentric spheres," by Lieut. A. Hamilton; "Multiply perfect numbers," by D. N. Lehmer.

At a meeting of the London Mathematical Society held on December 13, 1900, the following papers were read: Mr. A. B. Basset: "The real points of inflexion of a curve"; Miss M. E. Barwell: "On the conformal representation of polygons on a half plane"; Professor E B. Elliott: "The syzygetic theory of orthogonal binariants"; Mr. A. L. Dixon: "An addition theorem for hyperelliptic functions"; Professor W. Burnside: "On some properties of groups of odd order, II"; Mr. R. W. Hudson: "On discriminants and envelopes of surfaces"; Mr. H. W. Richmond: "Note on the inflexions of curves with double points."

The British association for the advancement of science will hold its next annual meeting at Glasgow, beginning on September 11. Professor A. W. Rücker is president of the association and Major P. A. MacMahon is president of the section of mathematical and physical sciences.

The Paris academy of sciences proposes the following questions for its mathematical prizes to be awarded at the annual session in December, 1902:—Grand prize of the mathematical section: To perfect in an important point the application of the theory of continuous groups to the study of partial differential equations.—Bordin prize: To develop and perfect the theory of surfaces applicable on the paraboloid of revolution.

The *Zeitschrift für Mathematik und Physik* (founded by Schlömilch) was transformed at the beginning of the present year into a journal for applied mathematics only, retaining, however, its former name. The editors are to be Professors R. MEHMKE, of Darmstadt, and C. RUNGE, of Hanover. There will also be an editorial committee consisting of Professors Bach, Hauck, Helmert, Klein, Lorentz, Müller, Linde, Seeliger and H. Weber.

UNIVERSITY OF GÖTTINGEN.—The following courses in mathematics are offered during the summer semester, 1901: —By Professor D. HILBERT: Algebra, four hours; Partial differential equations (with special regard to their applications), four hours; Seminar on partial differential equations, two hours.—By Professor F. KLEIN: Differential geometry, four hours, with seminar, two hours.—By Professor F. SCHILLING: Analytic geometry, four hours; Graphical statics and exercises.—By Dr. E. ZERMELO: Bessel's functions, two hours; Introduction for physicists, two hours.—By Dr. M. ABRAHAM: Theory of potential, two hours.—By Dr J. SOMMER: Determinants, one hour; Spherical trigonometry, two hours.

In applied mathematics:—By Professor G. BOHLMANN: Theory of probability, two hours.—By Professor W. VOIGT, Mathematical optics, four hours.—By Professor H. LORENZ, Technical thermodynamics, two hours.

PROFESSOR CHARLES HERMITE, the venerable dean of French mathematicians, died after a brief illness at his home in Paris the 14th day of January, 1901. He was born at Dieuze, December 25th, 1822. He entered the Ecole polytechnique in 1843, for which he was prepared at the lycée Louis-le-Grand. In 1848 he was made répétiteur of analysis and entrance examiner at the Ecole polytechnique; in 1863 he was appointed final examiner, and in 1869, professor of analysis in the same institution, in succession to Duhamel, whom he also succeeded as professor of algebra in the Faculty of sciences of Paris. In 1864 he was appointed maître de conférences at the Ecole normale. In 1856 he was elected to membership in the Paris academy of sciences as the successor of Binet.

MR. H. F. BAKER, F.R.S., has been appointed to a university lectureship in mathematics at Cambridge University.

AT the meeting of the Paris academy of sciences, December 31, 1900, Professor R. DEDEKIND, of the University of Bonn, was elected correspondent of the section of geometry.

MR. S. W. REAVES, graduate scholar in Cornell University, has been appointed instructor of mathematics at Orchard Lake Military Academy.

DR. C. N. LITTLE, professor of mathematics in Leland Stanford University, has resigned his position.

PROFESSOR G. BAUER, of the University of Munich, has been made emeritus professor of mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ASHTON (C. H.). Plane and solid analytic geometry; an elementary text-book. New York, Scribner, 1901. 12mo. 14 + 266 pp. Cloth. $1.00

BORTOLOTTI (E.). Lezioni di calcolo infinitesimale [dettate nell'anno accademico 1899–1900 nella r. università di] Modena. Modena, Pizzolotti, 1899–1900. 8vo. 621 pp.

BÜTTNER (F.). Ein Beitrag zur Theorie der Kugelfunktionen höherer Ordnung. 8vo. 38 pp. (From Festschrift zur 325-jährigen Jubelfeier des fürstl Stolberg'schen Gymnasiums zu Wernigerode, herausgegeben vom Lehrerkollegium der Anstalt, Leipzig, Fook, 1900.) M. 0.75

CARBONE (C.) Le congruenze del secondo ordine senza linee singolari e le loro superficie focali studiate mediante una trasformazione doppia. Catania, 1900. 8vo. 21 pp.

DOBROSERDOV (D. K.). See NERNST (W.).

FORSYTH (A. R.). Theory of functions of a complex variable. 2d edition. London, Clay, 1900. 8vo. 808 pp. 21s.

GAUSS (C. F) Allgemeine Flächentheorie (Disquisitiones generales circa superficies curvas). (1827.) Deutsch herausgegeben von A. Wangerin 2te Auflage. Leipzig, Engelmann, 1900. 8vo. 64 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 5.) M. 0.80

GULDBERG (A.). On partial differential equations of the third order. Christiania, Dybwad, 1900 8vo. 43 pp. (*Videnskabsselskabets Skrifter*, I, Math.-naturv. Klasse, No. 5.)

HALPHEN (G. H.). Traité des fonctions elliptiques et de leurs applications. Partie III: Fragments. (Quelques applications à l'algèbre et particulièrement à l'équation du 5e degré; quelques applications à la théorie des nombres; questions diverses.) Publié par les soins de la Section de Géométrie de l'Académie des Sciences. Paris, Gauthier-Villars, 1901 8vo. Fr. 8.50

KORN (A.). Lehrbuch der Potentialtheorie. Vol. II: Allgemeine Theorie des logarithmischen Potentials und der Potentialfunktionen in der Ebene. Berlin, Dümmler, 1900. 8vo. 10 + 366 pp. M. 9.00

LIEBMANN (H.). Lehrbuch der Differentialgleichungen. Leipzig, 1901. 8vo. 6 + 226 pp. M. 6.00

LÜROTH (J.). Vorlesungen über numerisches Rechnen. Leipzig, Teubner, 1900. 8vo. 7 + 194 pp. M. 8.00

MILLER (G. A.). On the groups which have the same group of isomorphisms. 1900. 4to. (*Transactions of the American Mathematical Society*, Vol. I, pp. 395–401.)

MOROT. Cours de calcul. Paris, Levé [1900]. 8vo. 79 pp.

NASILIEV (A. V.). See NERNST (W.).

NERNST (W.) and SCHÖNFLIES (A.). Short treatise on the differential and integral calculus for physicists, chemists, and naturalists. Translated into Russian from the second edition by D. K. Dobroserdov and A. V. Nasiliev. Moscow, 1901. 8vo. 15 + 351 pp. (Russian.) M. 6.00

SCHÖNFLIES (A.). See NERNST (W.).

SIMON (M.). Analytische Geometrie der Ebene. 2te Auflage. Leipzig, Göschen, 1900. 12mo. 207 pp. Cloth. (Sammlung Göschen, No. 65.) M. 0.80

WANGERIN (A.). See GAUSS (C. F.).

II. ELEMENTARY MATHEMATICS.

AHRENS (W.). Mathematische Unterhaltungen und Spiele. (In 2 Hälften.) Leipzig, Teubner, 1901. 8vo. 12 + 428 pp., 1 plate. Cloth. M. 10.00

AUGUST (E F). Vollständige logarithmische und trigonometrische Tafeln 23ste Auflage, besorgt von F. August. Leipzig, 1900. 12mo. 8 + 204 pp. Cloth. M. 1.60

BACKHAUS (K.). See WIESE (B.).

BARBISCH (H.). See JAHNE (J.).

BÉCHÉ (A.). See DUSSAUX (E.).

C. (F. I.). Table de logarithmes à cinq décimales pour les nombres de 1 à 10 000 et pour les fonctions trigonométriques de minute en minute. Edition stéréotype. Paris, Poussielgue [1900]. 16mo. 8 + 148 pp. (Cours de mathématiques élémentaires.)

CRACKNELL (A. G.). Practical mathematics. London, Longmans, 1900. 12mo. 376 pp. (Practical elementary science series.) 3s. 6d.

DUSSAUX (E.) et BÉCHÉ (A.). Première et deuxième années de géométrie dans l'enseignement primaire supérieur. Géométrie plane. Paris, Colin, 1900. 16mo. 344 pp. (Collection Julien Boitel.)

HARTL (H.). Die trigonometrische Auflösung des Dreieckes und der auf Dreiecke zurückzuführenden Figuren, für den Gebrauch an Werkmeister- und Baugewerkeschulen und für den Selbstunterricht bearbeitet. 2te Auflage. Mit 300 Uebungsbeispielen sammt Resultaten, nebst einer Tafel der Winkelfunktionen. Wien, Hölder, 1900. 8vo. 3 + 44 pp. Cloth. M. 0.96

HEGER (R.). Fünfstellige logarithmische, und goniometrische Tafeln, sowie Hülfstafeln zur Auflösung höherer numerischer Gleichungen, für den Gebrauch an höheren Schulen bearbeitet. Leipzig, Teubner, 1900. 8vo. 4 + 112 pp. Cloth. M. 1.60

HENRICI (J.) und TREUTLEIN (P.). Lehrbuch der Elementar-Geometrie (In 3 Teilen.) Teil III : Die Gebilde des körperlichen Raumes ; Abbildung von einer Ebene auf eine zweite (Kegelschnitte). 2te Auflage. Leipzig, Teubner, 1901. 8vo 12 + 192 pp. Cloth. M. 3 30

HOCHHEIM (A.). Leitfaden für den Unterricht in der Arithmetik und Algebra an höheren Lehranstalten. Heft 1. 6te Auflage, bearbeitet von F. Hochheim. Berlin, Mittler, 1900. 8vo. 6 + 258 pp. M. 3.00

JAHNE (J.) und BARBISCH (H.). Leitfaden der Geometrie und des geometrischen Zeichnens für Bürgerschulen Ausgabe in 1 Bande. Mit 320 Textfiguren, 156 geometrischen Ornamenten, einer Erklärungstafel für Bezirkspläne und zahlreichen Konstructions- und Rechenaufgaben. Wien, Manz, 1900. 8vo. 8 + 232 pp. Cloth. M. 2.40

——. Leitfaden der Geometrie und des geometrischen Zeichnens für Mädchen-Bürgerschulen. 1ste Stufe. Mit 78 Textfiguren und 140 geometrischen Ornamenten. Wien, Manz, 1900. 8vo. 4 + 56 pp. Cloth M. 0.90

JOHNSTON (N.). Johnston's geometrical division and measurement of arcs and angles. Bronson, Mich., Journal Press, 1900. 8vo. 10 pp. Cloth. $0.60

KODATIS (B.). Eine genauere und schnellere Berechnung des Kreises ohne Radiuswerth, nach neuen Gesichtspunkten aufgebaut. Berlin, 1900 8vo. M. 0.60

KÜHNE (H.) Mathematisch-technische Tabellen für den Schulgebrauch mit besonderer Rucksicht auf die Benutzung bei Prüfungen. Dortmund, Ruhfus, 1900. 8vo 48 pp. M. 1.50

LEBON (E.). Géométrie élémentaire, comprenant la géométrie plane et la géométrie dans l'espace, rédigée conformément au programme du brevet supérieur et des écoles normales primaires (cours de première et de deuxième année). Paris, Delalain [1900]. 16mo. 461 pp. Fr. 4.50

LICHTBLAU (W.). See WIESE (B.).

MAHLER (G.). Ebene Geometrie. 3te Auflage. Leipzig, Göschen 1900. 12mo. 158 pp. Cloth. (Sammlung Göschen, No. 41.) M. 0.80

MALLET (F.). Du rôle éducateur des mathématiques ; discours prononcé à la distribution des prix du petit séminaire et école libre du Sacré-Cœur d'Aix, le 26 juillet, 1900. Aix, Nicot, 1900. 8vo. 14 pp.

NESBITT (H. A.). Inductive geometry for transition classes. London, Sonnenschein, 1900. 12mo. 100 pp. Cloth. 1s. 6d.

ORMSBY (M. T.). Elementary practical mathematics. London, Spon, 1900. 8vo 422 pp. Cloth. 7s. 6d.

ORELLI (J.). Lehrbuch der Algebra fur Polytechnische und höhere Gewerbeschulen. 3te Auflage. Neuer Abdruck. (In 2 Banden.) Zurich, 1900. 8vo. M 10.00

RICH (A. W.). The new higher arithmetic Chicago, Flanagan Co [1900]. 12mo. 318 pp. Cloth. (Flanagan educational series.) $0.50

——. The new practical arithmetic; complete for all ordinary purposes. Chicago, Flanagan Co. [1900]. 12mo 222 pp. Cloth. (Flanagan educational series.) $0.75

ROBINSON (J. W). Robinsonian auto-calculator; an accountant's assistant, comprising multiplication and division tables, interest tables, etc Brookline, Mass, Robinson, 1900. Folio, 239 pp. Half cloth $5.00

RUPERT (W. W.) Famous geometrical theorems and problems, with their history (In 4 parts.) Part II. New York, Heath [1900]. 16mo pp. 27-58.

SCHUSTER (M.) Stereometrische Aufgaben. Ein Lehr- und Uebungsbuch zum Gebrauch beim Unterricht in den oberen Klassen höherer Schulen; mit besonderer Berücksichtigung der Methoden der darstellenden Geometrie bearbeitet. Leipzig, Teubner, 1900. 8vo. 7 + 80 pp, 1 plate. Cloth M. 1.40

TAYLOR (J. M.). Elements of algebra. Boston, Allyn & Bacon, 1900. 12mo. 14 + 461 pp. Cloth. $1.12

TREUTLEIN (P.). See HENRICI (J.).

WELLS (W.). Complete trigonometry; with answers. Boston, Heath, 1900. 12mo. 6 + 149 + 9 pp. Cloth. $1.08

——. Four-place logarithmic tables; together with a table of natural sines, cosines, tangents and cotangents Boston, Heath, 1900. 12mo. 1 + 23 pp Cloth. $0.25

WIESE (B), LICHTBLAU (W.) und BACKHAUS (K.) Raumlehre fur Lehrerseminare. Teil 2: Die Korperlehre (Stereometrie) und die wichtigsten Lehrsatze der Dreiecksrechnung (Trigonometrie). 3te Auflage. Breslau, Hirt, 1900. 8vo. 120 pp. Cloth. M. 1.65

WOLF (F C.). Praktische Geometrie fur den Schul- und Selbstunterricht; nach den Grundsätzen der Anschauung und Konzentration in genetischer Reihenfolge aufgebaut und unter besonderer Berucksichtigung der praktischen Bedürfnisse bearbeitet. Ausgabe fur Lehrer; mit Lösungen zu den Berechnungsaufgaben der Schulerhefte. Leipzig, Wunderlich, 1900. 8vo. 7 + 181 pp. M. 2.00

WROBEL (E.). Uebungsbuch zur Arithmetik und Algebra, enthaltend die Formeln, Lehrsätze und Auflösungsmethoden in systematischer Anordnung und eine grosse Anzahl von Fragen und Aufgaben. Zum Gebrauche an Gymnasien, Realgymnasien und anderen höheren Lehranstalten bearbeitet. Teil II: Pensum der Obersekunda und Prima des Gymnasiums. 3te Auflage. Rostock, Koch, 1900 8vo 4 + 164 pp Cloth. M. 1.60. Mit einem Anhang für hohere realistische Lehranstalten (Realgymnasien, Oberrealschulen u s. w.). 2te Auflage. 3 + 71 pp. M. 2.40

YOUNG (J W. A). The teaching of mathematics in the higher schools of Prussia New York, Longmans, 1900. 8vo 14 + 141 pp

III. APPLIED MATHEMATICS.

ADAMS (J. C) Lectures on the lunar theory, edited by R A Sampson. London and New York, Macmillan, 1900. 8vo. 88 pp. Cloth. (Cambridge University Press) $1 25

BELLAVITIS (E.). Prospettiva lineare insegnata nella scuola d'applicazione degli ingegneri di Padova. Padova, Prosperini, 1900. 8vo. 41 pp, 3 plates.

BERG (E J.). See STEINMETZ (C. P.)

BOCQUET (J. A.). Cours élémentaire de mécanique appliquée 4e édition, revue et augmentée Paris, 1900. 8vo. Cloth. Fr. 5 00

BÜTTNER (F.). Studien über die Green'sche Abhandlung: Mathematical investigations concerning the laws of the equilibrium of fluids (1832). Leipzig, Teubner, 1900. 8vo. 5 + 98 pp (Preisschriften gekrönt und herausgegeben von der Furstlich Jablonowski'schen Gesellschaft zu Leipzig. (XXXVI.) No XIV der mathematisch-naturwissenschaftlichen Section.) M. 6.40

CHADWICK (O.). See MIDDLETON (R. E.).

FARROW (F. R.). Stresses and strains. London, Whittaker, 1900. 12mo. Cloth. 5s.

HILDEBRANDSSON (H.) et TEISSERENC DE BORT (L). Les bases de la météorologie dynamique. Historique; état de nos connaissances (En 2 volumes.) IIe livraison, avec 20 planches. Paris, Gauthier-Villars, 1900. 8vo. Fr. 4 00

HOSKINS (L. M). Theoretical mechanics; an elementary text-book. Stanford University, Cal., published by the author, 1900 8vo 10 + 436 pp. Cloth. $3 25

JOHANNESSON (P.). Physikalische Mechanik. Berlin, 1900 8vo 58 pp., 2 plates. Boards. M 1 00

KRAFT und ENERGIE Eine kritische Betrachtung uber die Grundbegriffe der Mechanik. Wiesbaden, Bergmann, 1900. 8vo 6 + 65 pp M 1 30

LANZA (G.). Applied mechanics 8th revised edition. New York, Wiley, 1900. 8vo. 8 + 928 pp. Cloth. $7 50

MIDDLETON (R. E) and CHADWICK (O) Treatise on surveying, compiled as a text-book New York, Spon & Chamberlain, 1900 8vo 13 + 283 pp. Cloth $1 25

MONTI (C.). Corso di prospettiva teorico-pratica senza proiezioni ortogonali nè scala prospettica, d'avviamento alla copia dal vero, per gl'istituti tecnici, le scuole normali, tecniche, ecc. Torino, Paravia, 1900. 4to. 48 pp, 30 plates. Fr. 4 00

NICOLI (F.). Geometria descrittiva [lezioni tenute nella r. università di Modena nell'anno scolastico] 1899–1900. Modena, Pizzolotti, 1900. 8vo. 216 pp.

PILSWORTH (E. S.). A text-book on perspective. Battle Creek, Mich., School of Applied Art [1900]. 12mo. 2 + 52 pp. Cloth. $1.00

ROBERTS (H. A.). Treatise on elementary dynamics, dealing with relative motion mainly in two dimensions. London and New York, Macmillan, 1900. 12mo. 11 + 258 pp. Cloth. $1 10

SAMPSON (R. A.). See ADAMS (J. C.).

STEINMETZ (C. P.). Theory and calculation of alternating current phenomena. With the assistance of E. J. Berg. 3d edition, revised and enlarged. New York, Electrical World and Engineering Co., 1900. 8vo. 20 + 525 pp. Cloth. $4.00

TEISSERENC DE BORT (L.). See HILDEBRANDSSON (H.).

VOIGT (W.). Elementare Mechanik als Einleitung in das Studium der theoretischen Physik. 2te, umgearbeitete Auflage. Leipzig, Veit, 1901. 8vo. 10 + 878 pp. M. 14.00

WARE (W. R.). Modern perspective; a treatise upon the principles and practice of plane and cylindrical perspective. Revised edition. New York, Macmillan, 1900. 12mo. 8 + 336 pp. With plates in separate portfolio. Cloth. $4.00

WEISS (H.). Grundsätze der Kinematik. Heft I. Leipzig, 1900. 8vo. 1 – 256 pp., 10 plates folio. M. 10.00

WILSON (V. T.). Free-hand perspective; for use in manual-training schools and colleges. New York, Wiley, 1900. 8vo. 12 + 268 pp. Cloth. $2.50

ZECHLIN (M. R.). Kugel- und Rollenlager; Theorie, Berechnung und praktische Beispiele derselben. Berlin, Seydel, 1900. 8vo. 3 + 70 pp. M. 3.00

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THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE eighth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Thursday and Friday, December 27 and 28, 1900 There were two sessions each day, opening at 10 o'clock A. M. and 2.30 o'clock P. M. Thirty-eight persons were in attendance among whom were the following twenty-six members of the Society :

Professor Henry Benner, Dr. G. A. Bliss, Professor Oskar Bolza, Professor D. F. Campbell, Professor L. E. Dickson, Professor L. W. Dowling, Dr. J. C. Fields, Professor A. S. Hathaway, Professor Thomas F. Holgate, Dr. Kurt Laves, Professor Heinrich Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor H. B. Newson, Mrs. H. B. Newson, Professor D A. Rothrock, Dr. F. H. Safford, Professor Oscar Schmiedel, Miss Ida M. Schottenfels, Professor J. B. Shaw, Professor E. B. Skinner, Mr. Burke Smith, Professor E. J. Townsend, Professor C. A. Waldo, Dr. Jacob Westlund, Professor H. S. White.

Professor E. H. Moore, Vice-President of the Society, occupied the chair. The Christmas meeting being the regular time for the election of officers of the section, Professor Thomas F. Holgate was reëlected Secretary and Professors E. J. Townsend and J. B. Shaw were made members of the programme committee.

The following papers were read :

(1) Professor E. H. MOORE : "On the uniformity of continuity."

(2) Professor A. S. HATHAWAY : "Quaternions and fourfold space."

(3) Professor IRVING STRINGHAM : "On the geometry of planes in a parabolic space of four dimensions"

(4) DR. F. H. SAFFORD : "Flow of heat in two dimensions."

(5) MR. A. C. LUNN : "Certain mathematical aspects of experimental science."

(6) MR. E. A HOOK : "Some properties of circulating decimals"

(7) Professor ARNOLD EMCH : "Note on the congruences of twisted curves."

(8) Professor H B. NEWSON : "A generalization of the

Wessel-Gauss-Argand diagram" (preliminary communication).

(9) DR. F. R. MOULTON: "On straight line solutions of the problem of n bodies."

(10) DR. G. A. BLISS: "Geodesic lines on an anchor ring."

(11) MR. FRANZ A. LA MOTTE: "On the determination of the algebraic equations invariant under Tschirnhausen transformations, with the parameter representation of all such irreducible equations. with rational coefficients, of the third and fourth degrees."

(12) Professor E. J. TOWNSEND: "Functions of two real variables which are continuous with respect to each variable."

(13) Professor L. E. DICKSON: "The group of the equation for the twenty-seven lines on a general cubic surface."

(14) Professor OSKAR BOLZA: "Concerning the expression of abelian integrals in terms of a fundamental set of integral functions."

(15) Dr. J. C. FIELDS: "Proof of the Riemann-Roch theorem and of the independence of the conditions for adjointness."

(16) Professor OSCAR SCHMIEDEL: "Two reduction formulas applicable to certain particular integrals."

(17) Professor E. B. SKINNER: "Some forms which remain invariant with respect to certain ternary monomial substitution groups."

(18) Professor JAMES B. SHAW: "Note indicating a new development of a determinant."

(19) Professor E. H. MOORE: "On double limits."

(20) Professor E. H. MOORE: "Concerning the Harnack theory of improper definite integrals."

(21) Professor L. E. DICKSON: "Canonical forms of quaternary abelian substitutions in an arbitrary Galois field."

(22) Miss IDA M. SCHOTTENFELS: "Proof of the existence of a particular substitution group of degree twenty-one and order 20160."

Professor Hathaway also read a paper on "Pure mathematics for engineering students," which was followed by a very interesting discussion. This paper is printed in the present number of the BULLETIN.

Mr. Lunn was introduced to the Society by Dr. Moulton, Mr. Hook by Professor Skinner, Mr. La Motte by Professor Moore. Professor Stringham's paper was presented to the Society through Professor Moore and in the author's absence was read by Professor Hathaway; Professor Emch's paper and Professor Dickson's paper (No. 21) were read by title. Abstracts of the papers follow below.

In Professor Moore's paper on the uniformity of continuity, the well known theorem and a generalization of it which occurs in the theory of discontinuous functions are proved by a very simple method involving the partition of a fundamental interval (a, b) into three halves (a, d), (c, e), (d, b), where the points a c d e b are the extremities of the four quarters of (a, b).

The first part of Professor Hathaway's paper consisted of a development of quaternions from the definition: Quaternions is the most general linear associative algebra in which division is determinate. The merit claimed is in starting from this known property as a definition, and developing smoothly and briefly a working knowledge of quaternions, without reference to the units 1, i, j, k.

The characteristic equation of a number p is the equation of least degree with numerical coefficients that is satisfied by p, say,

$$p^{r+1} + c_1 p^r + \cdots + c_r p = 0.$$

There is one and only one such equation for a given number; it is not factorable into expressions of the same form; and c_r is not zero. This determines a unit

$$I = -(p^r + c_1 p^{r-1} + \cdots + c_{r-1} p)/c_r,$$

with the properties

$$I^2 = I, \quad Iq = qI = q,$$

where q is any quaternion. I is the scalar unit, and xI, or simply x, where x is a numerical coefficient, will denote a scalar number. A scalar number may also be defined as a quaternion whose square is a positive scalar; and a vector number is defined as a quaternion whose square is a negative scalar. Scalars are seen to be commutative factors.

Using scalar coefficients, the characteristic equation becomes, in irreducible form, $p^r + c_1 p^{r-1} + \cdots + c_r = 0$; and as this factors in the same way as ordinary equations with *real* coefficients, we must have $r = 1$ or $r = 2$, according as p is a scalar or a non-scalar. When p is a scalar, we take the square of its characteristic, and thus have a characteristic equation for any number p, of the form $(p - p_1)^2 + p_2^2 = 0$, where p_1, p_2 are scalars depending upon p. We define :

Scalar of $p = Sp = p_1$; vector of $p = Vp = p - p_1 = p - Sp$.

Tensor of $p = Tp = \sqrt{(p_1^2 + p_2^2)}$; unit or versor of $p = Up = p/Tp$.

Conjugate of $p = Kp = Sp - Vp$; angle of $p = \angle p = \cos^{-1} SUp$.

The characteristic equation of p is, in this notation,

$$(a) \qquad p^2 - 2Sp\ p + Tp^2 = 0.$$

Other forms of the same characteristic are

$$Tp^2 = pKp = Kp\ p = Sp^2 - Vp^2 = Sp^2 + TVp^2,$$
$$p^{-1} = Kp/Tp^2, \text{ etc.}$$

The uniqueness of the characteristic (a), determines the uniqueness of the resolutions

$$p = Sp + Vp = Tp \cdot Up = Up \cdot Tp, \text{ etc.}$$

From the characteristic equation of pq, we find by operating with $p^{-1}(\quad)p$, the characteristic of qp, and hence, $T(pq) = T(qp)$, $S(pq) = S(qp)$. In particular, if α, β be any two vectors we find

$$(\alpha\beta + \beta\alpha)(\alpha\beta - \beta\alpha) = (\alpha\beta)^2 - (\beta\alpha)^2 = 2S\alpha\beta(\alpha\beta - \beta\alpha) ;$$

and thence, whether $\alpha\beta = \beta\alpha$ or not, we find $\alpha\beta + \beta\alpha = 2S\alpha\beta$. Thus $(\alpha + \beta)^2$ is a scalar, whence $\alpha + \beta$ is either zero or a vector. Thence follow the usual distributive properties of the functional symbols S, V, K over a sum. Also, the conjugate of a product equals the product of the conjugates of its factors in reverse order ; the tensor of a product equals the product of the tensors of its factors, etc. The usual relations of products of vectors are found as in Tait, §§ 89–91 ; and in particular

$$\delta\ S(\alpha\beta\gamma) = \alpha S(\beta\gamma\delta) + \beta S(\gamma\alpha\delta) + \gamma S(\alpha\beta\delta);$$

or, given three independent vectors, any fourth vector can be expressed in terms of them. If there is only one independent vector, the system is the ordinary imaginary system. If there are two unit vectors α, β that are independent, then $\alpha - \beta$, $\alpha + \beta$, $(\alpha - \beta)(\alpha + \beta) = 2V\alpha\beta$ are three independent vectors, whose units are, say i, j, k, with the relations $i^2 = j^2 = k^2 = -1 = ijk$.

The second part of the paper is an application of quaternions to fourfold space, first given in the BULLETIN, November, 1897, and afterwards extended in the papers on Alternate processes (*Proceedings of the Indiana Academy of Sciences*, 1897) and Linear transformations in four dimensions (abstract in BULLETIN, November, 1898, no. 5, p. 93). The application is founded on the definitions

(1) Line $(w + xi + yj + zk) =$ line whose components along the four mutually perpendicular axes of reference are w, x, y, z;

(2) Line p + line q = line $(p + q)$,

(3) Line $p \cdot$ line q = line pq.

With p as multiplier and q as multiplicand there are two products pq and qp. Such multiplications *with* or *by* p, are the two kinds of multiplication referred to as direct and contra multiplication in the first paper, and are now called in multiplication and by-multiplication. A given in- or by-multiplication is shown to possess a system of invariant planes, one and only one through each line; and the angular displacement in each plane is constant in magnitude and therefore in sense, and equal to the angle of the multiplier. The ratio of elongation is the tensor of the multiplier. The invariant planes of an in-multiplication are called in-parallel planes; and by-parallel planes are invariant planes of a by-multiplication. A given angularly-directed plane is shown to be a plane of an in-parallel system of one and only one unit vector α, and a plane of a by-parallel system of one and only one unit vector β; and these vector units (corresponding to directions in ordinary or vector space) are called the in direction and by direction of the given plane. In the case of two planes, the angle between these in-directions is called their in-angle of inclination, and the angle between their by-directions their by-angle of inclination. It is shown that these angles remain unaltered by rigid displacement, the effect of such rigid displacement being to give all in-directions of planes a revolution about one vector axis, and all by-directions a revolution about another vector axis. It is shown that two planes have in general two and only two common perpendiculars (a perpendicular to a plane being a plane that "transverses" it in a right diedral angle). These two common perpendiculars are "orthogonal" to each other, *i. e.*, every line of one is perpendicular to every line of the other. In-parallel or by-parallel planes are shown to have a continuous turn-parallel system of orthogonal common perpendiculars, and all "plane" angles between the two lines of intersection with a common perpendicular are equal. For any two planes the two plane angles of intersection by the orthogonal common perpendiculars are the half sum and half difference of the in-angle and by-angle between the planes. The area-projecting factor between two planes is in sign and magnitude the product of the cosines of their plane angles. The second part of this paper will appear in the *Transactions*.

Professor Stringham adopts essentially definitions (1) and (2) of Professor Hathaway's paper, but not definition (3). The subject is analytically treated by equations of loci, with the interpretation and manipulation of their parameters. The fundamental equation of a plane through the origin is $\alpha r + r\beta = 0$ where α, β are unit vectors and r is the director line of any point R of the plane. This is the plane which Professor Hathaway fixes as of in-direction $\pm\alpha$ and by-direction $\mp\beta$. After developing the analytical material of distances between points, and points and planes, and of angles between lines, the author discusses the maximum angle between two variable lines lying in given planes, applying calculus methods to the varying parameters. He finds in general two such angles called the plane angles of the two planes. He finds the projective factor between the two planes $\{\alpha, \beta\}$, $\{\alpha', \beta'\}$ to be $-\frac{1}{2}(S\alpha\alpha' + S\beta\beta')$, which is also the product of the cosines of the plane angles. This expression is Professor Hathaway's

$$\tfrac{1}{2}(\cos\theta + \cos\varphi) = \cos\tfrac{1}{2}(\theta + \varphi)\cos\tfrac{1}{2}(\theta - \varphi).$$

Professor Stringham defines this projective factor as the cosine of the angle between the planes. This paper is to be published in the *Transactions*.

Dr. Safford inserts a term expressing surface leakage in the usual differential equation for the flow of heat in two dimensions, considers the possibility of obtaining solutions of a particular form for this equation, and deduces the isothermal curves and curves of flow. The methods used involve the discussion of a solution of a partial differential equation in the form λ, R_1, R_2, where R_1 and R_2 are functions of the two independent variables respectively, and are the general solutions of two ordinary differential equations. The factor λ is a function of both variables but does not contain the integration constants appearing in R_1 and R_2.

Mr. Lunn showed that if a physical hypothesis is given by implication in the form of an equation defining the value of a variable quantity as a function of the time, there exists at least a double infinity of functions which are not distinguishable from it by measurement. and so far to be considered as equally verified by observation. Also if there exists one satisfactory hypothesis, there exists an infinity. Hence an induction using only the observed values and no hypothetical elements can never give a definite result.

Mr. Hook gave a method by which the number of digits in the repetend of any circulating decimal may be found when the prime factors of the denominator are known. His result differs from similar results found in text books in that it gives the exact number of digits while most such formulas yield only a number of which this is a divisor.

In Professor Emch's paper a study is made of the congruence of curves of intersection of the surfaces $F(x, y, z, a, b) = 0$ and $\varphi(x, y, z, a, b) = 0$. If $b = f(a)$ the system of curves lies upon a surface which has an envelope when

$$\frac{\partial F}{\partial a} + \frac{\partial F}{\partial b} \cdot \frac{db}{da} = 0 \text{ and } \frac{\partial \varphi}{\partial a} + \frac{\partial \varphi}{\partial b} \cdot \frac{db}{da} = 0.$$

Eliminating x, y, z from these conditions and the equations of the surfaces, a new relation

$$\psi\left(a, b, \frac{db}{da}\right) = 0$$

is found. If this last equation has a singular solution, the corresponding system of curves forms the singular surface of the congruence and all surfaces of the congruence are tangent to it.

In Dr. Moulton's paper those solutions of the problem of n bodies are sought in which the bodies always lie in a straight line and describe conic sections with respect to their center of gravity. The problem was solved by Lagrange for three bodies; this paper reduces the problem in the general case of n bodies to the solution of simultaneous algebraic equations. The solutions, which can be obtained in each case for a particular value of one of the masses, are studied as functions of this mass regarded as variable; it is shown that these functions have no pôles and, by the non-vanishing of the Jacobian of the functions with respect to the coördinates, that they have no branch points. Therefore there are $\frac{1}{2}n!$ distinct straight line solutions of the problem of n bodies for all values of the masses. The method applies with only slight modifications for laws of force varying as any power of the distance.

Dr. Bliss discussed the forms of the geodesic lines on the anchor ring. The curves of minimum length were found by the methods of the calculus of variations and the resulting equations, which define a doubly infinite system of geodesic lines, were expressed in terms of Weierstrassian σ, ζ, and $\wp$ functions. By applying the properties of these functions the shapes of the geodesic lines are determined. In the paper all points of the surface of this ring are classified. The paper is intended for publication in the *Transactions.*

Mr. La Motte's paper is in abstract as follows: Given a realm of rationality Ω and in Ω an equation $f(x) = 0$ of the nth degree. This latter is invariant under Tschirnhausen transformations of the same realm if, and only if, its Galois group in Ω is a subgroup of one or more of certain groups which as to type are fully determined for each degree n. By means of resolvents a general method is developed to find parameter representations of all *irreducible* equations whose Galois groups are subgroups of one of the groups mentioned. In the cases of the third and fourth degrees these parameter representations are actually found. They represent for every set of rational parameters a required equation of the given degree, and also every required equation is represented by one of them and a set of rational parameters. These formulæ for the fourth degree are

$$x^4 + px^3 + qx^2 + rx + s =$$

$$(1) \qquad x^4 + \frac{4\vartheta(1-\xi) + \eta^2}{1 - \xi + \varepsilon(\eta - \varepsilon\vartheta)}(\varepsilon x^3 + \xi x^2 + \eta x + \vartheta) = 0$$

and

$$(2) \qquad x^4 + \frac{4p_2 - p_3^2}{p_1}\left(x^3 + \frac{p_2}{p_1}x^2 + \frac{p_3}{p_1}x + \frac{1}{p_1}\right) = 0,$$

with the added negative conditions 1° that the discriminant of

$$(3) \qquad t^3 - qt^2 + (pr - 4s)t - (r^2 - 4qs + p^2 s) = 0$$

does not vanish, and 2° that none of the three numbers

$$\sqrt{p^2 - 4q + 4\sigma_i} \qquad (i = 1, 2, 3)$$

is rational, where σ_i $(i = 1, 2, 3)$ are the roots of (3). For the third degree we have the simple formula

$$x^3 + qx + r = x^3 - \frac{1 + 27\varepsilon^2}{4\vartheta^2}\left(x + \frac{\varepsilon}{\vartheta}\right) = 0,$$

with the negative condition that

$$4q^3 + 27r^2 \neq 0.$$

In these formulas the Greek letters denote parameters which are, as the case may be, arbitrary or definite quantities of the realm Ω.

Professor Townsend's paper dealt with the applications of the double limit to the investigation of certain properties of a function defined as follows: Let $f(x, y)$ be a one-valued function of two real independent variables x and y and continuous with respect to each variable separately within the region, $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_n$. Then it follows, that within this region the regular points (*i. e.*, points where $\underset{\substack{x=a\\y=b}}{L} f(x, y) = f(a, b)$) must be everywhere dense, but at the same time the irregular points (*i. e.*, where either $\underset{\substack{x=a\\y=b}}{L} f(x, y)$ does not exist or is different from $f(a, b)$) may be also everywhere dense. When further $f(x, y)$ is defined on the boundary $y = y_0$ for each value of $x = x_0$ by the limit $\underset{y=y_0}{L} f(x_0, y) = f(x_0, y_0)$, then it follows that $f(x, y_0)$ can be discontinuous in x at a set of points everywhere dense in the interval (α, β), but not at every point of this interval. By aid of this result it was shown that the points on the boundary $y = y_0$ at which $f(x, y)$ is continuous with respect to both variables together must lie everywhere dense. When all of the points of the boundary, including the end points of the interval, are regular points, then we have the necessary and sufficient condition for the uniform convergence of the function $f(x, y)$ toward the boundary function $f(x, y_0)$. These same results hold concerning the convergence of an infinite series of continuous functions of x, since the convergence of such a series is but a special case of the convergence of the function defined above toward the boundary function $f(x, y_0)$. The necessary and sufficient condition that $f(x, y_0)$ and hence that a function defined by an infinite series of continuous functions be continuous at $x = x_0$ is that there exists a set of values of y, say $y = a_1, a_2, a_3, \cdots, a_k, \cdots$, dense at y_0, such that for each such value of a_k we have $| f(x, y_0) - f(x, a_k) | < \sigma$ for every value of x within a definitely defined interval $(x_0 - \delta_{a_k}, x_0 + \delta_{a_k}')$, which may, however, vary with a_k and may have, in fact, an under limit equal to zero, thus differing from the necessary and sufficient condition for uniform convergence. The full paper from which this is taken constituted the author's dissertation for the doctor's degree and has already been published.

Professor Dickson's paper (No. 13), outlined a chapter of his work on linear groups which is soon to appear in Teubner's Sammlung. An orthogonal group O of order 25920 on 5 indices modulo 3 is readily shown to be isomorphic with the abelian linear group for the trisection of the periods of a hyperelliptic function of four periods. A rectangular table for O having 27 rows serves to define an isomorphic substitution group which is recognized as a subgroup of index two of the group of the equation for the 27 lines on a general cubic surface. This proof of the identity of the two problems avoids the lengthy calculations of M. Jordan, the discoverer of the relation.

The object of Professor Bolza's paper* is to prove by methods of the theory of functions the converse of the propositions given in Baker's work on abelian functions, chapter IV, concerning the expression of abelian integrals in terms of a fundamental set of integral functions.

Let $f(x, y) = 0$ be an irreducible equation of degree n in y; suppose that for a finite value $x = a$ there are m branch places $(a, b_1), (a, b_2), \cdots, (a, b_m)$ of orders $w_1, w_2, \cdots, w_m$ respectively, where $(w_1 + 1) + (w_2 + 1) + \cdots + (w_m + 1) = n$, and put, in the vicinity of (a, b_k), $x - a = t_k^{w_k + 1}$.

(*a*) If then $g_0, g_1, \cdots, g_{n-1}$ constitute a fundamental set of integral functions for $f(x, y) = 0$, and if the expansion of g_i in (a, b_k) is

$$g_i = \sum_{\nu=0}^{\infty} c_{k\nu}^{(i)} t_k^{\nu}$$

the determinant

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* After the paper was finished, I found that I had been anticipated in essential points, by Landsberg, Zur Algebra des Riemann-Roch'schen Satzes, *Math. Annalen*, vol. 50, p. 333. O. BOLZA.

$$D = \left|c_{k\nu}^{(0)} c_{k\nu}^{(1)} \cdots c_{k\nu}^{(n-1)}\right|, \qquad (\nu = 0, \cdots, w_k;\ k = 1, 2, \cdots, m),$$

is different from zero.

(*b*) Let now $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ denote the complementary functions associated with $g_0, g_1, \cdots, g_{n-1}$; then it follows from the characteristic property* of the complementary functions, that in the vicinity of (a, b_j)

$$\sum_{\iota=0}^{n-1} \sum_{\mu=0}^{\infty} c^{(\iota)}_{k,\, \rho+\mu(w_k+1)}(x-a)^{\mu}\gamma_{\iota} = \frac{\delta_{jk}}{(w_j+1)t_j^{\rho}}$$

where $\delta_{jk} = 1$ or 0 according as $j = k$ or $\neq k$, for $\rho = 0, 1, \cdots, w_k$; $k = 1, 2, \cdots, n$. The determinant of these n equations with $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ as unknown quantities is an ordinary power series in $(x - a)$ which for $x = a$ reduces to the non-vanishing determinant D. Thence follows

1. In the vicinity of (a, b_j)

$$\gamma_{\iota} = \frac{P(t_j)}{t_j^{w_j}} \qquad (j = 1, 2, \cdots, m),$$

P being the general symbol for an ordinary power series.

2. If $\nu = \rho + q(w_k + 1)$, where $0 \leqq \rho \leqq w_k$, and if we define

$$\gamma_{k\nu} = \sum_{\mu=0}^{q} \sum_{\iota=0}^{n-1} c^{(\iota)}_{k,\, \rho+\mu(w_k+1)} (x-a)^{\mu}\gamma_{\iota},$$

this function γ_k has in the vicinity of (a, b_j) the expansion

$$\frac{\gamma_{k\nu}}{(x-a)^q} = \frac{\delta_{jk}t_j^{-\nu}}{w_j+1} + t_j P(t_j).$$

(*c*) If, in particular, $g_0 = 1, g_1, \cdots, g_{n-1}$ form a *normal* fundamental set ("normal basis") of dimensions $d_0 = 1, d_1, \cdots, d_{n-1}$ respectively, the expansions of the functions γ_{ι} and $\gamma_{k\nu}$ at infinity can be obtained by combining the above results with Weber's theorem according to which in this case

$$h_0 = \frac{g_0}{x^{d_0}},\ h_1 = \frac{g_1}{x^{d_1}},\ \cdots,\ h_{n-1} = \frac{g_{n-1}}{x^{d_{n-1}}}$$

constitute a normal fundamental set for the algebraic equation

$$f\left(\frac{1}{\xi}, y\right) = 0.$$

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* Baker, l. c., p. 63.

The result is that

1. The integrals

$$\int x^{d_i - i}\,\gamma_i\,dx \qquad (i = 1, 2, \cdots, n-1)$$

are integrals of the first kind.

2. The integral

$$\int \frac{-\,{}^{\nu}\gamma_{k\nu}\,dx}{(x-a)^{q+1}}$$

is an integral of the second kind with the one pole (a, b_k) and the expansion

$$\frac{1}{t_k^{\nu}} + P(t_k)$$

in its vicinity.

3. The integral

$$\int \left[\frac{\sum\limits_{i=0}^{n-1} g_i(\xi, \eta)\gamma_i(x, y)}{x - \xi} - \frac{\sum\limits_{i=0}^{n-1} g_i(\xi', \eta')\gamma_i(x, y)}{x - \xi'} \right] dx$$

is an elementary integral of the third kind with the logarithmic points (ξ, η) and (ξ', η').

Dr. Field's proof of the Riemann-Roch theorem is as follows: Let $F(z, u) = 0$ be the equation to an irreducible algebraic curve of degree n. The form of this equation is supposed to be such that the only multiple points are double points which are not at the same time branch points. Regard u as the dependent variable and further assume that no line parallel to the axis of u passes through more than one double point, or is tangent at a double point, or is an asymptote, and also that the asymptotes are all distinct from one another and no two parallel to each other. To an equation of the character in question, the equation of any irreducible algebraic curve can be reduced by a birational transformation.

By $(a_1, b_1), \cdots, (a_d, b_d)$ indicate the d double points, and by $(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$ any other q points of the curve, and consider the form of the function

$$(1) \qquad \sum_{\lambda=1}^{d+q} \frac{\gamma_\lambda F'(a_\lambda, u)}{(z - a_\lambda)\,(u - b_\lambda)} + T(z, u),$$

where $T(z, u)$ is an arbitrary polynomial in (z, u) and the quantities γ_λ are arbitrary constants. This form represents the most general rational function of (z, u) which, apart from infinities at ∞, only becomes infinite for points among the q points

$$(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$$

and for these to the first order only.

By a purely algebraic process seek to determine the limitations which must be imposed upon the form (1) in order that it may represent a function which is finite at ∞. We find that the most general rational function whose infinities are all included under the q infinities here in question is given by the form

$$(2) \qquad \sum_{\lambda=1}^{d+q} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + c,$$

where c is an arbitrary constant and the coefficients γ_λ satisfy the system of $\frac{1}{2}(n-1)(n-2)$ equations

$$(3) \qquad \sum_{\lambda=1}^{d+q} \gamma_\lambda a_\lambda^i b_\lambda^k = 0, \qquad (i + k = 0, 1, 2, \cdots, n-3).$$

In the case $q = 0$ the form (1), for arbitrary values of the coefficients γ_λ, represents the most general rational function of (z, u) which becomes infinite only at ∞, and the form (2), for conditioned values of the coefficients γ_λ furnished by the equations (3), represents the most general rational function which is nowhere infinite.

Now, an algebraic function which is nowhere infinite can only be a constant, and in the case in question—namely $q = 0$—it follows that the coefficients γ_λ in (2) must all have the value 0. The system of equations

$$(4) \qquad \sum_{\lambda=1}^{d} \gamma_\lambda a_\lambda^i b_\lambda^k, \quad (i + k = 0, 1, 2, \cdots, n-3)$$

can therefore only be satisfied when the quantities γ_λ all have the value 0.

From the interpretation of the equations (4), it immediately follows that the d conditions for adjointness in the case of a curve of degree $n - 3$, and therefore also in the case of a curve of higher degree, are independent of one another.

In the general case $q + 0$, an examination of the system of equations (3) shows that the number of arbitrary quantities γ_λ involved in their solutions is $q - s$, where s is the "*strength*" of the system of q points $(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$ in determining an adjoint polynomial of degree $n - 3$. From (2) it is seen that $q - s + 1$ is the number of arbitrary constants involved in the expression of the most general rational function of (z, u), whose infinities, all of the first order, correspond to points among the q points here in question.

Professor Schmiedel presented the results of an investigation into the general formula of reduction for the function

$$\int x^m y^n dx, \quad \text{where} \quad y = \sum_0^{t} t a_t x^t,$$

and deduced two formulas by which the reduction of the exponents m and n may be effected.

The forms studied by Professor Skinner are such that they are invariant with respect to substitutions of the form $z_1' = a_1 z_i$, $z_2' = a_2 z_j$, $z_3' = a_3 z_k$, $(i, j, k, = 1, 2, 3$ in some order), where a_1, a_2, a_3 are roots of unity and $a_1 a_2 a_3 = \pm 1$. The groups for which i, j, k have the order 1, 2, 3 are abelian and all such ternary monomial groups can be generated by at most two independent generators. If T_1, T_2 denote the two generators of the group and τ a multiplicative substitution of order 2 with determinant $- 1$, all ternary monomial groups may be found by combining the substitutions T_1, T_2, τ with the generators of the symmetric group of three elements. The systems of invariant forms are then found by setting up the forms which are invariant with respect to the generators of the symmetric group, taken singly or together and subjecting these forms to the substitutions T_1, T_2, τ. The conditions for invariance appear in the form of certain linear homogeneous congruences, the modulus being either the order of T_1, or of T_2. The computation of the reduced systems and the complete systems depends largely upon auxiliary quantities which appear in the solution of the congruences.

Professor Shaw exhibited a method for the development of any determinant in terms of partial determinants of the general form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & 0 & 0 & a_{44} & \cdots & a_{4n} \\ a_{51} & 0 & 0 & 0 & \cdots & a_{5n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & 0 & 0 & & a_{nn} \end{vmatrix}$$

The central theorem of Professor Moore's paper (No. 19) on double limits, which will appear in the *Transactions*, will be sufficiently indicated by the following particular case. On the supposition that a function $f(x, y)$ is defined for

$$-1 < x < 1, \quad -1 < y < 1, \quad x \neq 0, \quad y \neq 0$$

and that the limits

$$\underset{x=0}{\mathrm{L}} f(x, y), \quad \underset{y=0}{\mathrm{L}} f(x, y),$$

exist and are respectively denoted by

$$h(y); \quad g(x)$$

for respectively

$$-1 < y < 1, \quad y \neq 0; \quad -1 < x < 1, \quad x \neq 0,$$

the necessary and sufficient condition that the limits

$$\underset{y=0}{\mathrm{L}}\, h(y), \quad \underset{x=0}{\mathrm{L}}\, g(x)$$

exist and are equal is that $f(x, y)$ approaches its limit $h(y)$ on the y-set $-1 < y < 1$, $y \neq 0$, subuniformly near $y = 0$, that is, that for every positive ε there exists a positive δ_ε ($\delta_\varepsilon \leqq 1$) and for every x, ($x \neq 0$, $|x| < \delta_\varepsilon$) there is a positive $\delta_{\varepsilon x}'(\delta_{\varepsilon x}' \leqq 1)$ such that

$$|f(x, y) - h(y)| < \varepsilon$$

for every (x, y) satisfying the conditions

$$|x| < \delta_\varepsilon, \quad |y| < \delta_\varepsilon', \quad x \neq 0, \quad y \neq 0.$$

An equivalent condition is that $f(x, y)$ approaches its limit $g(x)$ in an analogous way.

The definition and the exposition of the elements of the theory of improper definite integrals as given by Harnack in volumes 21 and 24 (1883, 1884) of the *Mathematische Annalen*, leave much to be desired. In recent memoirs in the *Wiener Sitzungsberichte* (1898, 1899), and in the appendix to the third volume of his treatise on the calculus (1899) Stolz has given a new development of the theory, in which the absolutely convergent integrals play the leading rôle. Professor Moore's paper (No. 20), written in the spirit of the original Harnack memoirs, contains a development of the theory throwing additional light on the varying properties of the absolutely and the conditionally convergent integrals. This paper will be published in the *Transactions*.

Professor Dickson's second paper (No. 21) is also intended for publication in the *Transactions*. A set of canonical forms for the substitutions of a given group should possess the property that two of its substitutions are conjugate within the group, if, and only if, they are reducible to the same canonical form according to a definite scheme of reduction. For linear groups in a Galois field of order p^n, reduction to a canonical form belonging to a field of order p^{nr}, $r > 1$, must be possible by the introduction of new indices conjugate with respect to the initial $GF[p^n]$. The memoir gives a set of canonical forms belonging to the abelian group itself and deduces a set of ultimate canonical forms, the former depending upon the coefficients of the characteristic equation, the latter upon its roots. By these results tables are derived which give a classification into conjugate sets of all quaternary abelian substitutions and also of the operators of a single quotient group. For $p^n = 3$, the latter is the group of order 25920 occurring in the problem of the 27 lines on a general cubic surface. Its operators fall into 20 sets of conjugates: the identity, one set of period 5, 4 sets of period 3, 2 sets of each of the periods 2, 4, 9 and 12, and 6 sets of period 6.

Miss Schottenfels presented a proof of the existence of a substitution group of degree 21 and order 20160, having cyclical subgroups identical with those enumerated by Professor Dickson in the *American Journal of Mathematics*, vol. 22, p. 252, in the ternary linear fractional group of order 20160 in the Galois field $[2^2]$.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL

INDIRECT CIRCULAR TRANSFORMATIONS AND MIXED GROUPS.

BY PROFESSOR H. B NEWSON.

(Read before the American Mathematical Society, February 23, 1901.)

1. IN a paper[*] entitled "Continuous groups of circular transformations," the writer gave a list of the continuous groups of circular transformations in the plane and their chief properties. That paper treated only of direct transformations The present paper is supplementary to the former, and deals with indirect circular transformations and the mixed groups obtained by combining these with the direct transformations.

I. PROPERTIES OF INDIRECT CIRCULAR TRANSFORMATIONS.

2. Certain fundamental properties of indirect circular transformations were developed by Fricke and published[†] in 1890. I have seen no paper of more recent date dealing with the subject. Let T be the symbol of a direct, and $\overline{T}$ the symbol of an indirect, circular transformation. Fricke's results, which form the starting point of this paper, may be stated as follows:

The second power of $\overline{T}$ is a direct circular transformation, which is either hyperbolic, parabolic or elliptic. $\overline{T}$ leaves invariant two real points, one real point, or no real points, according as its second power is hyperbolic, parabolic, or elliptic. There are three varieties of indirect transformations, viz., the hyperbolic, parabolic, and elliptic, designated by $\overline{hT}$, $\overline{pT}$ and $\overline{eT}$, respectively, distinguished according to the character of their second powers and hence also according to the number of their invariant points. $\overline{hT}$ and $\overline{hT}^2$ leave invariant the same pair of invariant points A and A'; $\overline{pT}$ and $\overline{pT}^2$ have the same invariant point A; $\overline{eT}$ has no invariant point, but interchanges the pair of points which $\overline{eT}^2$ leaves invariant. An indirect transformation of period two is an inversion of the plane with respect to a real or pure imaginary circle. Other properties of $\overline{T}$ will now be developed.

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[*] BULLETIN (2), vol. 4, pp. 107–131 (Dec., 1897).

[†] Klein-Fricke's Modulfunctionen, vol. 1, pp. 196–207.

3. Since $\overline{hT}$ transforms circles into circles and leaves invariant a pair of points A, A', it must transform into itself the hyperbolic system of circles through A and A'; since $\overline{hT}^2$ leaves invariant every circle of the system through A and A', one effect of $\overline{hT}$ must be to interchange in pairs the circles of this hyperbolic system. Such an interchange can be effected only by an inversion of the plane with respect to $\overline{C}$, one of the circles of the system. From these conditions it follows that an indirect transformation $\overline{hT}$ must be the resultant of a direct transformation hT and an inversion I on one of the path curves of hT.

The inversion I leaves invariant every point on $\overline{C}$ and every circle orthogonal to $\overline{C}$; it also produces an involutoric transformation, $k = -1$, in every circle orthogonal to $\overline{C}$. The hyperbolic system of circles through A and A' contains one and only one circle C orthogonal to $\overline{C}$. Since $\overline{C}$ and C remain invariant under both hT and I, they must also be invariant under their resultant $\overline{hT}$. Hence, *an indirect hyperbolic transformation* $\overline{hT}$ *leaves invariant a pair of points* (AA') *and a pair of orthogonal circles intersecting in* A *and* A'.

If k be the cross ratio of the direct transformation hT, then $\overline{hT}$ produces along the circles $\overline{C}$ and C one dimensional hyperbolic transformations characterized by k and $-k$ respectively. Two circles through A and A' making equal angles with $\overline{C}$ are interchanged by $\overline{hT}$. Thus the second power of $\overline{hT}$ leaves invariant all circles through A and A' and is therefore a direct transformation of the hyperbolic kind.

There are evidently ∞^1 indirect transformations $\overline{hT}$, one for each real value of k, each of which has the same fundamental invariant figure. When $k = 1$, $\overline{hT}$ reduces to the inversion with respect to $\overline{C}$; when $k = -1$, it becomes the inversion with respect to C.

4. By a course of reasoning similar to the above we infer the following properties of an indirect parabolic transformation: The effect of an indirect parabolic transformation $\overline{pT}$ on the points of the plane is equivalent to a direct parabolic transformation with constant a, whereby all points in the plane are moved along their path curves, followed by an inversion with respect to $\overline{C}$, one of these path curves. Hence $\overline{pT}$ leaves invariant a single point A and one circle $\overline{C}$ through A. Along $\overline{C}$ it produces a one dimensional para-

bolic transformation whose constant is a. Two circles touching $\overline{C}$ at A and making equal angles with $\overline{C}$ are interchanged. There is evidently a different transformation on the same invariant figure for each real value of a. The transformation of this system for which $a = 0$ is the inversion of the plane with respect to $\overline{C}$. Thus, *an indirect parabolic transformation* $\overline{pT}$ *leaves invariant a single point* A *and a circle through* A *and produces along the circle a one dimensional parabolic transformation.*

5. Similarly we see that an indirect elliptic transformation $\overline{eT}$ is the resultant of a direct elliptic transformation eT and an inversion on one of its path curves. eT leaves invariant two points A and A' and moves all points in the plane along circular path curves around A and A'. Such a transformation followed by an inversion on $\overline{C}$, one of these path curves, results in $\overline{eT}$. A and A' being inverse points with respect to every path curve of the system are interchanged by $\overline{eT}$; so that $\overline{C}$ is the only invariant figure of $\overline{eT}$. The path curves within $\overline{C}$ are interchanged with those without $\overline{C}$ in pairs. $\overline{eT}$ produces along $\overline{C}$ an elliptic transformation with cross ratio $e^{i\theta}$. There are ∞^1 indirect elliptic transformations leaving $\overline{C}$ invariant and interchanging A and A', one for each value of θ. The transformation corresponding to $\theta = 0$ is the inversion on $\overline{C}$.

The circle $\overline{C}$ is a path curve of eT and hence may be either real or a pure imaginary circle, *i. e.*, without real points. A real inversion may take place about an imaginary circle as well as about a real circle. In the hyperbolic and parabolic cases inversion about an imaginary path curve does not occur for the reason that every path curve has on it at least one real point. Thus we see that *an indirect elliptic transformation* $\overline{eT}$ *leaves invariant a single circle and produces along that circle a one dimensional elliptic transformation.*

Our discussion may be summed up in the following theorem:

Theorem 1. Every indirect circular transformation of the plane is the resultant of a direct non-loxodromic transformation and an inversion on one of its path curves.

II. Mixed Groups of Circular Transformations.

6. The Mixed Group mG_6. The indirect circular transformations whose properties have just been developed do

not form continuous groups, but taken with direct transformation form mixed groups. There are ∞^6 direct and ∞^6 indirect circular transformations of the plane and the aggregate of all these forms the mixed group mG_6. This general group contains many subgroups which are also mixed groups; these we now proceed to examine.

7. THE MIXED GROUP $mG_2(AA')$. We seek the aggregate of all transformations both direct and indirect which leave two points A and A' separately invariant, and also those which interchange them. Two points A and A' remain invariant under ∞^2 direct transformations which form the continuous group $G_2(AA')$. Let P and Q be two points harmonic to A and A' (whence $PQAA'$ are four points on a circle). The involutoric transformation in the group $G_2(PQ)$ interchanges A and A'. Since ∞^2 pairs of points may be chosen harmonic to A and A', we see that there are ∞^2 direct involutoric transformations interchanging A and A'.

The indirect transformations which leave A and A' separately invariant are all of the hyperbolic variety. There are ∞^1 circles through A and A'; each of these with the invariant points A and A' is the invariant figure of ∞^1 indirect hyperbolic transformations. Thus we see that there are ∞^2 indirect hyperbolic transformations leaving A and A' separately invariant.

The indirect transformations which interchange A and A' are all of the elliptic variety. There are ∞^1 circles in the elliptic system of which A and A' are the vanishing points. Each of these circles is invariant under ∞^1 indirect elliptic transformations which interchange A and A'. Thus we see there are ∞^2 indirect elliptic transformations interchanging A and A'.

The aggregate of all direct and indirect transformations which leave a point pair invariant forms a mixed group $mG_2(AA')$. From the above discussion we see that it contains two kinds of direct and two kinds of indirect transformations. The decomposition of $mG_2(AA')$ into subgroups will be discussed in articles 10 and 11.

Theorem 2. *The mixed group $mG_2(AA')$ contains ∞^2 direct transformations (which form the continuous group $G_2(AA')$), ∞^2 direct involutoric transformations, ∞^2 indirect hyperbolic, and ∞^2 indirect elliptic transformations.*

8. THE MIXED GROUP $mpG_3(A)$. We seek next the aggregate of all parabolic transformations both direct and indirect which leave a single point A invariant. We know that A remains invariant under ∞^2 direct parabolic transformations which form the continuous group $G_2(A)$. The

indirect transformations which leave a single point A invariant are all of the parabolic type. There are ∞^2 circles through A; each of these circles with the invariant point A is the invariant figure of ∞^1 indirect parabolic transformations. Hence there are ∞^3 indirect parabolic transformations leaving the point A invariant. The aggregate of all parabolic transformations both direct and indirect which leave A invariant forms a mixed group, $mpG_3(A)$. Its subgroups will be investigated in article 12.

Theorem 3. *The mixed group $mpG_3(A)$ contains ∞^3 indirect and only ∞^2 direct parabolic transformations.*

9. THE MIXED GROUP $mG_4(A)$. We know that there are ∞^4 direct transformations leaving a single point invariant and these form a continuous group $G_4(A)$. There are also ∞^4 indirect hyperbolic transformations having one invariant point at A; the second invariant point A' is in turn every point in the plane. There are also ∞^3 indirect parabolic transformations leaving A invariant. The aggregate of all these transformations both direct and indirect forms a mixed group $mG_4(A)$. This mixed group contains no indirect elliptic transformations.

Theorem 4. *The mixed group $mG_4(A)$ contains ∞^4 direct, ∞^4 indirect hyperbolic, and ∞^3 indirect parabolic transformations.*

10. THE MIXED GROUP $mhG_1(AA'\overline{C})$. The continuous group $G_2(AA')$ contains one continuous subgroup of hyperbolic transformations $hG_1(AA')$, whose invariant path curves are the circles of the hyperbolic system determined by A and A'. Each of these circles passing through A and A' is invariant under ∞^1 indirect hyperbolic transformations. Let $\overline{C}$ be one of these circles; then $\overline{C}$ is invariant under ∞^1 direct and also ∞^1 indirect hyperbolic transformations. These two systems form the mixed group $mhG_1(AA'\overline{C})$.

Within the mixed group $mG_2(AA')$ there are ∞^1 such subgroups, one for each circle of the hyperbolic system determined by A and A'. All of these mixed groups contain the same continuous group of direct hyperbolic transformations $hG_1(AA')$.

Theorem 5. *The mixed group $mG_2(AA')$ contains ∞^1 mixed subgroups $mhG_1(AA'\overline{C})$, all of which contain the same continuous group $hG_1(AA')$.*

11. THE MIXED GROUP $meG_1(AA'\overline{C})$. The continuous group $G_2(AA')$ contains one continuous subgroup of elliptic transformations $eG_1(AA')$ whose invariant path curves are the ∞^1 circles having A and A' for a pair of inverse points.

Each circle of this elliptic system is invariant under ∞^1 indirect elliptic transformations. Let $\bar{C}$ be one of these circles, then $\bar{C}$ is invariant under ∞^1 direct and also ∞^1 indirect elliptic transformations. These two systems form the mixed group $meG_1(AA'\bar{C})$.

Within the mixed group $mG_2(AA')$ there are ∞^1 such subgroups, one for each circle of the elliptic system determined by A and A'. Each of these mixed groups contains the same continuous group of elliptic transformations, $eG_1(AA')$.

Theorem 6. *The mixed group $mG_2(AA')$ contains ∞^1 mixed subgroups $meG_1(AA'\bar{C})$, all of which contain the same continuous group $eG_1(AA')$.*

12. THE MIXED GROUPS $mpG_2(A)$ AND $mpG_1(A\bar{C})$. The continuous group $pG_2(A)$ contains ∞^1 continuous groups $pG_1(A)$. The path curves of one of these subgroups form a parabolic system of circles through A. Each circle $\bar{C}$ of this parabolic system is invariant under ∞^1 indirect parabolic transformations; consequently the circle $\bar{C}$ and the point A on it are invariant under ∞^1 direct and also ∞^1 indirect parabolic transformations, which therefore form a mixed group $mpG_1(A\bar{C})$. The group $mpG_3(A)$ contains ∞^2 such groups $mpG_1(A\bar{C})$, one for each circle through A.

The aggregate of all the groups $mpG_1(A\bar{C})$ whose invariant circles form a parabolic system of circles through A constitute a group $mpG_2(A)$. For this parabolic system of circles remains invariant under all of these ∞^2 transformations; the system as a whole remains invariant though the individual circles of the system are not necessarily invariant under all the transformations of the group. The ∞^1 subgroups $mpG_1(A\bar{C})$ of $mpG_2(A)$ all contain the same continuous subgroups $pG_1(A)$.

Theorem 7. *The mixed group $mpG_3(A)$ contains ∞^1 mixed subgroups $mpG_2(A)$ and each of these contains ∞^1 mixed subgroups $mpG_1(A\bar{C})$. All subgroups $mpG_1(A\bar{C})$ of $mpG_2(A)$ contain the same continuous subgroup $pG_1(A)$.*

13. THE MIXED GROUP $mG_3(\bar{C})$. Let $\bar{C}$ be any real circle in the plane; we know that $\bar{C}$ is invariant under ∞^3 direct transformations which form a continuous group $G_3(\bar{C})$. Every direct transformation T in this continuous group followed by an inversion on the circle $\bar{C}$ results in an indirect transformation $\bar{T}$, and the aggregate of all these transformations both direct and indirect forms a mixed group $mG_3(\bar{C})$.

Every pair of points on $\overline{C}$ are the invariant points of a mixed group of hyperbolic transformations $mhG_1(AA'\overline{C})$, leaving $\overline{C}$ invariant; hence $mG_3(\overline{C})$ contains ∞^2 such mixed groups of hyperbolic transformations. Every point on $\overline{C}$ is the invariant point of a mixed group of parabolic transformations $mpG_1(A\overline{C})$, leaving $\overline{C}$ invariant; hence $mG_3(\overline{C})$ contains ∞^1 such mixed subgroups of parabolic transformations. Every pair of points inverse with respect to $\overline{C}$ are interchanged or remain invariant under the transformations of a mixed elliptic group $meG_1(AA'\overline{C})$; hence $mG_3(\overline{C})$ contains ∞^2 such mixed subgroups of elliptic transformations.

Theorem 8. *The mixed group* $mG_3(\overline{C})$ *contains* ∞^2 *mixed subgroups* $mhG_1(AA'\overline{C})$, ∞^2 *mixed subgroups* $meG_1(AA'\overline{C})$, *and* ∞^1 *mixed subgroups* $mpG_1(A\overline{C})$.

14. THE MIXED GROUP $mhG_2(A\overline{C})$. The mixed group $mG_3(\overline{C})$ contains ∞^2 hyperbolic subgroups $mhG_1(AA'\overline{C})$. Let the point A remain fixed and let the point A' be in turn every point on $\overline{C}$. The aggregate of all the subgroups $mhG_1(AA'\overline{C})$ whose invariant points satisfy this condition forms a mixed group $mhG_2(A\overline{C})$. This group contains one parabolic subgroup $mpG_1(A\overline{C})$, but no elliptic subgroups. The group $mG_3(\overline{C})$ contains ∞^1 such subgroups $mpG_2(A\overline{C})$, one for each point on the circle $\overline{C}$.

Theorem 9. *The group* $mG_3(\overline{C})$ *containing* ∞^1 *subgroups* $mhG_2(A\overline{C})$, *each of which contains hyperbolic and parabolic transformations but no elliptic transformations.*

15. THE MIXED GROUP $mG_3(i\overline{C})$. Let $i\overline{C}$ be any imaginary circle in the plane. We know that $i\overline{C}$ is invariant under ∞^3 direct elliptic transformations which form the continuous group $G_3(i\overline{C})$. Every transformation in this group followed by an inversion on the circle $i\overline{C}$ results in an indirect elliptic transformation $\overline{eT}$ and the aggregate of all these transformations both direct and indirect forms the mixed group $mG_3(i\overline{C})$. This group evidently contains ∞^2 mixed subgroups $meG_1(AA'\overline{C})$, one for each pair of points in the plane inverse with respect to the circle $i\overline{C}$.

Theorem 10. *The group* $mG_3(i\overline{C})$ *contains only elliptic transformations; these fall into* ∞^2 *subgroups* $meG_1(AA'\overline{C})$.

Résumé.—We have thus found a list of eleven mixed groups of circular transformations of the plane. These with their appropriate symbols and invariant figures are as follows:

1. mG_6. No invariant figure.
2. $mG_4(A)$. A single invariant point.
3. $mpG_3(A)$. A single invariant point.
4. $mG_3(\bar{C})$. A real invariant circle.
5. $mG_3(i\bar{C})$. An imaginary invariant circle.
6. $mpG_2(A)$. A single invariant point.
7. $mG_2(AA')$. An invariant point pair.
8. $mhG_2(A\bar{C})$. An invariant point and a circle through it.
9. $mhG_1(AA'\bar{C})$. A pair of invariant points and a pair of orthogonal circles.
10. $mpG_1(A\bar{C})$. An invariant point and an invariant circle.
11. $meG_1(AA'C)$. An invariant point pair and an invariant circle.

LAWRENCE KANSAS,
January 18, 1901.

PURE MATHEMATICS FOR ENGINEERING STUDENTS.

BY PROFESSOR A S. HATHAWAY.

(Read before the Chicago Section of the American Mathematical Society, December 28, 1900.)

I. ITS UTILITY.

I HAVE had opportunity to become acquainted with the written opinions of graduates of from one to many years' standing, with regard to the benefit and utility of their instruction in all departments of technical work, and have also taken the opportunity of conferring personally with graduates in respect to my own department. In discussing this subject, I shall therefore appear, not as a special pleader for pure mathematics, but as one who proposes to present its claims in their true proportions to the other necessary work of the student.

In the first place, mathematical analysis is not so directly useful to the average engineer as the mathematician might expect. The mathematics that an engineer is obliged to use regularly is of that cut and dried form which is found tabulated in engineering handbooks of easy access, so that only a little arithmetic or algebra, and occasionally some trigonometry, answer all practical purposes. Seventy-five per cent. of graduates of technical institutes, now in actual practice, have probably forgotten how to differentiate and integrate, although most of them could easily refresh their memory in this respect if it were necessary. We can say, in favor of the calculus, that the remaining twenty-five per cent. include those who are leaders in their profession, and those who occupy positions under which new conditions are constantly arising and in which originality of thought is required; and among these are many who have found use for all the mathematics that they could acquire.

The question arises, when so few engineering students afterwards make practical use of the higher mathematics, Is it advisable to teach it thoroughly to all students? The answer that I find to this question is, yes. Almost every practical engineer who has been properly taught, sees the benefit of such studies. Such engineers will generally grant of their own accord the practical benefits of their mathematical training, even though they have had no occasion to use the analysis as an instrument of investigation. Many wish that they could use it better, as they see opportunities where it might be of advantage. if it were at their fingers' ends; and all agree that, aside from its practical use, there is a training in habits of thought, points of view, and intellectual comprehension of ordinary engineering problems, which only the study of the higher mathematics can give. The fact is that the fundamental ideas of natural phenomena that appear in all technical work are found in their most logical and clearest form in the higher mathematical analysis. The development of the mind in logical relations whose analogues exist in some form or other in nature, creates that mental insight which perceives the dominant ideas and develops the most practical methods of treatment of the problems of each day.

The technical training that is involved in the study of the higher mathematics, particularly the calculus, therefore answers the question as to its value and utility for all engineering students, irrespective of any direct value of the analysis as an instrument of investigation.

II. METHODS OF INSTRUCTION.

Some practical men who look upon mathematical analysis only as an instrument of the engineer, would have it taught solely for the purpose of obtaining the requisite mechanical skill in its use. It is to be made a matter of memorizing formulas and methods, while any explanation is recommended that placates the mental comprehension of the student, whether it is logical and common sense or not, so long as it is brief. This is not only depriving the majority of students of the real benefits that accrue to them from the study of mathematics, such as in the acquirement of fundamental ideas of technical value, but it is also a dangerous method of instruction. When a young engineer can do something supposed to be of value in his profession, but about which he has only certain vague conceptions and no perfect understanding from a common sense point of view, it is quite natural that he should give it an exaggerated importance ; and by an equally obscure mental process, he obtains an exalted opinion of his own capabilities. The result is frequently disastrous to himself in some important work that he may have been led to undertake in the fancy that he was equal to it. The frequency of such cases, as coming from the graduates of a certain school, is certain evidence of faulty methods of instruction, particularly in mathematics. One difference between rigorous and logical methods of instruction, and memory training, is in their effect on what is called the personal equation of the student, *i. e.*, his liability to make mistakes. The first methods bring such equation before the student, show that he is in the habit of superficial thinking and of readily accepting plausible conclusions that are more often wrong than right. He is forced to develop a system of checks upon his ideas and conclusions. and to exercise a continual care in discovering and eradicating the effects of those lapses of mental acumen to which all are more or less liable according to the training of the individual and the temporary condition of the mind. On the other hand, training by memory alone leaves the individual in ignorance of his unconscious errors of thought, and while he may be very expert in his lessons, yet in the elements of detailed and accurate work, he may prove nothing but a blunderer, with all actual mental training to begin over again in practical life if he is to succeed.

Mathematical analysis should therefore be taught according to the precepts of good logic and common sense. It is not necessary to consider all the detailed refinement of the pure mathematician, but some of the consequences of that refinement should not be neglected, and above all the student should learn that some very good methods of demonstration for ordinary cases do have their exceptions. A combination of the recitation system, based upon a good text book, with the lecture system seems best, both because it secures that flexibility of course hereafter considered, and because a proper amount of lecture work is in itself a training in quickness of apprehension on the part of the student, and creates the ability to seize upon the important parts of a subject. In fact, extensive note-taking of lectures on the part of the student should not be encouraged, the idea being to work up the subject afterwards from as few notes as possible.

The working of examples, especially numerical ones, cannot be too excessive, if within the time of the student. In this connection, most students can, by proper thought given to methods of study and work, reduce the time necessary to be spent on that subject by fully one-half the usual amount spent unmethodically. This means, perhaps more time spent when beginning a subject, in learning thoroughly its fundamental principles and methods, and less time later on—just the reverse, it will be seen, of the usual procedure of the average student.

III. The Course.

With reference to the framework of a technical mathematical course, I should leave it as it is in most institutes—algebra, geometry, trigonometry, analytical geometry, the calculus, and the elements of differential equations with particular reference to those equations occurring in engineering subjects. Under geometry should be included the elements of projective geometry, especially of parallel projection, with applications to the conic sections. The calculus cannot be begun too early, before analytic geometry even. The calculus is the fundamental topic; its ideas are so useful in mental development that when they are thoroughly mastered, other subjects are more easily acquired, so that it is in reality a time-saver. With us also, the last term of mathematical instruction is given up to the student's own devising in the way of subjects and development. Each student is expected to prepare what is essentially a course of lectures on chosen topics before the class. His aids in this work are whatever suggestions and information he can acquire from sources within his reach—the library, text books, his teachers. The result has been an improvement on knowledge already gained, and a development of pure and applied mathematics on the part of the student far beyond the usual course. Remembering that each student selects his own line of work, and hears, and is naturally interested in, whatever his classmates present, I have come to consider this term's work as the most valuable of all.

A course should be made flexible, by introducing at suitable times and places subjects that are interesting and likely to prove valuable in practical work. All pure mathematics is valuable as mental discipline; but the adaptability of different kinds will change with the years, the class, the character of the students, and the particular interests that they may be led by circumstances to manifest. For instance, I had, from a recent junior class, a request for a course in determinants, which is only briefly considered under algebra. The reason for this request was that the professor in dynamical engineering was at the time considering the subject of transformer problems, such as the problem of distribution of alternating current over a long line having capacity with up and down transformers at either end, and had found the determinant notation extremely useful in this work.

As another illustration of the changes which time brings, take the case of the applications of the imaginary analysis. The continuous current has had its day, although the practical applications of Faraday's discovery in 1832 was delayed nearly half a century by the want of a practical method of converting the alternating current into a continuous one. Recent discoveries have made the alternating current the more valuable in practical applications; and in this current we find a beautiful analogue of the imaginary analysis—so much so that technical writers are using that analysis in their publications on alternating currents for practical engineers. The imaginary analysis should therefore be taught to-day in live technical schools, with special reference to its geometrical and physical interpretations.

Instances might be multiplied showing that new circumstances will continually arise that necessitate deviations from old and established lines of instruction. Such flexibility in the course can only be secured, without continual change in its framework, through the lecture system in connection with text book instruction.

IV. The Instructor.

The success of a course, in mental training, and in meeting the practical needs of the engineer, depends very largely upon the instructor. He must be one who keeps up an interest in pure mathematics and their practical applications, and who is also clear in his own ideas, so that he may know what is interesting and useful, and be able to introduce it with clearness and brevity when occasion offers. He should, in other words, be a high grade man—not the highest, for that is impossible for all schools to secure, but certainly of such reputation among pure as well as applied mathematicians as causes him to be regarded with respect by the leaders in those subjects.

There is an objection in certain technical schools to a mathematical teacher who has too much fondness for pure mathematics. I once heard the president of such a school not only express this objection, but go further, by saying that he did not want a man in his faculty who knew more than himself on any subject.

The principal cause of the above objection to higher mathematical knowledge seems to be a fear that it leads to talking over the heads of the students. I will contend that it is an advantage to a school to have instructors who are able to talk over the heads of their students, and who do it occasionally. When an instructor talks over his head, the student may have some very strong feelings as to the futility of the exhibition, but he will find that the ideas do not always fly over. Like seeds planted in soil they await the germinating time. There is an unconscious impression made upon the mind by listening to ideas that are from a higher point of view than our own, and such impressions are often more durable than those left by apprehended impressions, and finally, as the result of gradual ascent to the higher point of view, they become fully understood and appreciated.

It is of course not possible to rely upon the results of such influences for the instruction of a technical student, and some ability to make himself understood and bring his mind within the confined horizon of the student, is necessary in the occupant of the instructor's chair. Such an instructor who also attempts to lift the students above their too small horizon at the certain risk of talking sometimes over their heads, produces, for some reason, students of higher acquisition, and of greater enthusiasm and practical ability, than is possible for the drill master to accomplish with his rule of thumb. It takes time to determine the influence of a teacher, and in the end the conscientious student is bound to recognize the superior influence upon his mental development of the instructor of superior attainments.

ROSE POLYTECHNIC INSTITUTE.

ADAMS'S UNPUBLISHED PAPERS.

The Scientific Papers of John Couch Adams, Vol. II. Part I., *Extracts from Unpublished Manuscripts*, edited by R. A. SAMPSON. Part II., *Terrestrial Magnetism*, edited by W. G. ADAMS. Cambridge, Pitt Press, 1900. 4to. xxxii + 646 pp., with 6 magnetic charts.

Lectures on the Lunar Theory. By JOHN COUCH ADAMS, edited by R. A. SAMPSON. Cambridge, Pitt Press, 1900. 8vo. 88 pp.

BY those who knew John Couch Adams best the examination of the manuscripts which he left and the publication of a selection from them has been looked forward to with much interest. His well known reluctance to publish anything not in a complete form—"I have some finishing touches to put to it"—raised hopes that valuable results might be contained in the packets of neatly-written and dated papers which were turned over after his death for examination. It was known that he had been at work on various problems, in particular on that most difficult one—the theory of Jupiter's satellites, and on the theory of terrestrial magnetism. It was known, too, that Adams rarely attacked a problem without throwing a new light upon it or evolving some unexpected result. He had not fallen into the modern habit of making "preliminary communications," or, at least, if these were made they were verbal and did not find their way into print. Thus the publication of this volume was looked forward to with greater expectation than is usual in the case of other scientists who have lived to his age.

I shall not be misunderstood in saying that a slight feeling of disappointment arises on turning over its pages and seeing the matter contained therein. Adams's reputation as a mathematician and astronomer of the highest class rests on too firm a basis to be disturbed by anything which may or may not be contained in his unpublished papers. His published work, small in quantity though it may be, made an ineffaceable mark on the subjects which he touched. The discovery of Neptune, the determination of the accurate value of the secular acceleration of the moon's mean motion, the method by which he arrived at the correct period of the November meteors are sufficient to stamp the character of his work without mentioning other results which show equal ability but are not so well known. If we are disappointed at finding no great discovery in the new volume or anything indicating that he was on the track of one, we have no reason to regret its appearance or to feel that the labor and thought which the editors have expended in putting the manuscripts into form have been in any sense wasted. There is, in fact, almost nothing which is not new. Some of the papers consist of new methods in handling old problems, others consist of the detailed work in the solution of problems, the mere results of which had alone been published, and, more important than all these, his great work on terrestrial magnetism is set forth in a complete and connected form.

The first number consists of a selection from Adams's lectures on the lunar theory. These have also been published separately without change for the use of those to whom the larger volume might not be easily accessible. The editor, Professor R. A. Sampson, has not aimed at giving the lectures in the complete form in which Adams delivered them. Portions which contained alternative proofs or explanations which are to be found in the text books have been omitted. The result is a continuous development which gives the coefficients of the principal inequalities due to the sun's action with a good degree of approximation. In his preface, Professor Sampson says "Of current elementary theories it may be said that they leave off where the difficulties of the subject begin, that is to say, where the various cases of slow convergence have been exposed, but not dealt with. It is perhaps not too much to say that these lectures carry us to the point where such difficulties end, in an adequate evaluation of all the chief constants. They leave the problem effectively solved and not merely stated, and show the path clear for the formation of a detailed theory, if that is desired." This is hardly correct as to the question of convergence, or at any rate it obscures the issue. The difficulty of slow convergence can hardly be said to have been dealt with by Adams any more than by his predecessors. That he has obtained close approximations is due to the fact that the numerical value of m,* the ratio of the mean motions, is used instead of series proceeding in powers of m. There is not much difficulty about convergence in the case of the moon when the numerical value of m is used at the outset, since slow convergence only occurs along powers of m. Again, one of the chief difficulties of the subject is the calculation of the higher terms with sufficient accuracy. Owing to small divisors the earlier terms have to be calculated with far greater accuracy than would be necessary if these small divisors did not arise, and this difficulty has not been bridged in any other way than by calculating the earlier terms to a high degree of accuracy.

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* More accurately, he uses $m/(1-m)$.

The method used by Adams is practically the same as that of G. W. Hill, except that he has adopted polar where Hill uses rectangular coördinates. For the earlier inequalities this method works well enough, but it would most probably give rise to unmanageable calculations for the higher ones if a complete theory were in contemplation. In general, it lacks the elegance and capacity for high approximation with a given amount of labor which characterizes Hill's methods, with one or two exceptions, notably that by which Adams arrives at the equation from which the motion of the perigee is obtained. In fact, the symmetry possible with rectangular coördinates disappears altogether when polar coördinates are used, and in the latter case the expansions of such expressions as

$$\cos\,(a_0 + a_1 \cos t + a_2 \cos 2t + \cdots)$$

have to be continually performed while, in the former, we have only to do with multiplications of algebraic series.

The next paper contains the detailed work necessary for the expression of the infinite determinant giving the motions of the node or perigee. In No. 3, entitled "Numerical developments in the lunar theory," we have the numerical values of the coefficients of δn when we put $n + \delta n$ for n and $n' + \delta n'$ for n' with $\delta n = \delta n'$ in the variational inequalities. The values of the coefficients of the parallactic inequalities are also obtained to 16 places of decimals, the first power of the ratio of the parallaxes only being retained. The determination of the part of the secular acceleration of the moon's mean motion which depends on m only is found in the next paper, and in No. 15 without having recourse to expansions in powers of m. Adams's method is in reality a numerical one and depends on the changes in the values of the coefficients of several periodic terms, the principal ones being those found in the previous paper. The writer has shown how this part of the secular acceleration can be found from the *constant* term in the parallax with any degree of accuracy required.*

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* *Proc. Lond. Math. Soc.*, 1896.

From these papers it appears that Adams had calculated the following sets of inequalities with the accuracy necessary for a complete lunar theory: The parts of the variational terms which depend on m only and those which depend on m and the second power of the solar eccentricity; the terms depending on m and the first powers of the solar eccentricity, the ratio of the parallaxes, and the inclination; and the principal part of the motion of the node. Other inequalities, *e. g.*, the terms depending on m and the first power of the lunar eccentricity had been found to a less degree of accuracy; some of these appear in the lectures.

A short note on Neison's lunar inequality due to Jupiter follows. Adams finds the theoretical coefficient of this with some accuracy by considering it as a sort of Jovian evection, thus making use of the literal expression for the solar evection. The value obtained differs by about 15 per cent. from that of M. Radau,* who has calculated a large number of the planetary inequalities.

No. 6 contains a novel suggestion for the solution of the equation

$$\frac{d^2w}{dt^2} + (a_0 + a_1 \cos 2t + a_2 \cos 4t + \cdots)w = 0$$

which occurs so frequently in celestial mechanics. The solution is of the form

$$w = \Sigma A_i \cos \{(k + 2i)t + a\},$$

and the difficulty of calculation arises from the presence of small divisors due to the fact that k is in general very nearly equal to unity. Adams's method really amounts to finding all the other coefficients in terms of k, A_{-1} in such a way that no small divisors occur; the two equations for these quantities are solved as a final step.

Papers 7, 8, 9, 12 are on Jupiter's satellites. As Professor Sampson has presented Adams's investigations, it appears that a new method of treating the whole subject was in contemplation, somewhat analogous to that which he used for the motion of the moon. He first solves the problem of the motion of a satellite about an oblate primary, in fact he finds the inequalities due to the figure of Jupiter. The equations for the three coördinates are reduced to two simultaneous ones of the second order with the distance and perpendicular on the equatorial plane as dependent variables and the time as independent variable. The equations are solved literally so that they can be applied to any satellite. The third coördinate is then easily found, as the editor shows in a short note. The development of

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* *Ann. de l' Observ. de Paris*, vol. 21.

$$[a^2 - 2aa' \cos (nt - n't + \varepsilon - \varepsilon') + a'^2)]^{-\frac{1}{2}},$$

necessary to obtain the mutual actions of the satellites on one another, is then made with some completeness and applied, for illustration, to the first two satellites. The mutual perturbations of two satellites when the eccentricities and inclinations are neglected next follow. Here again polar coördinates are used so that the perturbations are found directly, instead of indirectly by the more usual method of the variation of arbitrary constants. Values for the radius vector and the longitude, of the proper form, are substituted in the differential equations and the coefficients are determined by equations of condition.

As Adams does not go beyond the first order of the disturbing forces in the published investigations, the difficulty arising from the exact relation

$$n - 3n' + 2n'' = 0,$$

where n, n', n'' are the mean motions of the first three satellites, does not come into consideration. In fact, as far as the theory is given, we have simply those inequalities which are analogous to the variational and parallactic terms in the lunar theory. Some numerical results are given. It would be interesting to see how this method of procedure would work out when applied to attack the complete problem. Perhaps Adams did not see his way clear for this and therefore refrained from publishing what he had already done. The chief difficulties undoubtedly arise mainly in the terms depending on the eccentricities and on the square of the disturbing forces owing chiefly to the smallness of $n - 2n'$, $n' - 2n''$ and to the exact equality of these two expressions. The other papers on this subject contain corrections of the masses of the satellites and of errors in Damoiseau's and Laplace's theories.

Every astronomer knows of H. A. Newton's investigations on the orbit of the November meteors in which he arrived at the conclusion that the observations could in general be satisfied by any one of five competing values for the mean period. Adams supplemented this by showing that only one of these five values was consistent with the observed motion of the node of the orbit. In paper 10 we have the details of the method by which he arrived at the result. The brevity of the work is remarkable. The calculations are not given in detail. They are reduced to mechanical quadratures and a few of the principal steps for the perturbations by Jupiter are set forth. The method is a modification of Gauss's formulæ.

Some extracts from Adams's lectures on the Figure of the Earth occupy No. 11. They consist of theorems on the attractions of spheres and spheroids when the law of force is any function of the distance and conclude with the well-known differential equation connecting the ellipticity of strata with the density. A couple of notes on the analytical interpretation of Newton's methods for finding the variation and motion of the apse, a short method for obtaining certain of Laplace's expansions of functions in power series, and lists of errata in the works of Plana, Damoiseau, and de Pontécoulant occupy the remaining pages of Part I. In the last there are one or two slight errors. On p. 236, line 3, for 1025, read 1024; omit line 5, which is given on the last line but one of the previous page; on line 7 insert a + before the last number.

The second part, containing Adams's work on terrestrial magnetism, occupies two-thirds of the volume. The difficulties of Professor W. G. Adams, who undertook to edit this portion of the work left by his brother, must have been sufficiently great The matter had not been put into final form, though it was probably ready for it. The editor's chief labor was to gather together the various packets of theoretical and numerical work; to discover their connection, and by blending them to set forth in a complete manner the results of Adams's labors. A glance at the memoir shows that this task can have been no light one, and students will be grateful to Professor W. G. Adams for the manner in which he has performed it. In a preface he gives a summary which shows well the nature of the work and the conclusions reached by his brother.

The method adopted is that of Gauss, with modifications. But instead of the 24 constants with which Gauss contented himself, Adams has included 240, just ten times as many. Moreover, the observations which in Gauss's time were few and badly distributed, have increased much in volume, though still far from what is really necessary for a thorough determination. The memoir is arranged in eight sections, of which the first four consist of various relations between the harmonics and coefficients used in such investigations. Sections V, VI, respectively treat of the theory of terrestrial magnetism when the earth is considered as a sphere and as a spheroid. The last two sections contain the numerical calculations. At the end are maps drawn from the numerical results for every 5° of latitude and every 10° of longitude. The agreement with observation is stated by the editor to be very satisfactory.

In conclusion, the reader will gather from the rough indications here given that the volume is not a mere compilation of incomplete fragments. Whatever he may think about the value of collections of published papers, and in particular of Volume I of Adams's works, it will be immediately apparent that Volume II is at least on the same footing as any other book containing original and previously unpublished scientific work, and its possession by scientific libraries will be as much a necessity as is that of the best known journals and treatises.

Ernest W. Brown.

NOTICE SUR M. HERMITE.

PAR M. C. JORDAN.

ADDRESS DELIVERED AT THE MEETING OF THE PARIS ACADEMY OF SCIENCES, JANUARY 21, 1901.

The French school of mathematics loses in the person of M. Hermite its head and master.

It would be rash to undertake to analyze in haste and under the stress of keen emotion the long series of his works which have thrown so much lustre on the second half of the nineteenth century. Such an undertaking calls for more time and calmer feelings. Addressing then to our venerated confrère the last farewell, which his modesty forbade pronouncing at his grave, we limit ourselves here to pointing out, in broad lines and as far as memory permits, some of the discoveries which we owe to him.

In 1843, M. Hermite entered the Ecole Polytechnique at the age of twenty years. At the suggestion of Liouville, he wrote to Jacobi communicating the results which he had obtained relative to the division of abelian functions, then but little known. The illustrious German geometer, who was occupied at the time with the editing of his works, did not hesitate to give the letter of his young correspondent a place beside his own investigations. He wrote to him a little later: "Do not be troubled, Monsieur, if some of your discoveries are to be found in my former researches. As you commenced where I finish, there is necessarily a small sphere of contact. In the future, if you honor me with your communications, I only shall have to learn."

The prediction of the great geometer was not long in being fulfilled. In the four letters which follow and which Jacobi has also preserved for us, M. Hermite proposed first to generalize the theory of continuous functions; but he shortly found himself led to vaster problems in the arithmetical theory of forms, where he soon obtained admirable results. At the very beginning of his works he points out several methods for reducing quadratic forms in any number of indeterminates. A little later the introduction of continuous variables leads him to the discovery of deeper lying truths.

He gives the complete solution of the problem of the arithmetical equivalence of general quadratic forms or of forms decomposable into linear factors; he determines the transformations of these forms into themselves; he demonstrates, in a manner altogether novel and purely arithmetical, the celebrated theorems of Sturm and Cauchy on the separation of the roots of algebraic equations. He introduces the fertile notion of quadratic forms with conjugate variables, and deduces from their theory a new demonstration of the beautiful theorem of Jacobi on the number of decompositions of a number into four squares. He arrives finally at the remarkable proposition that the roots of all algebraic equations having integral coefficients and the same discriminant can be expressed by a limited number of distinct irrationalities.

The algebraic study of forms is also the object of his meditations. The notion of invariant which dominates this theory had remained a little confused until the day when Cayley threw it into full light in the celebrated memoir dated 1845. Cayley, Sylvester, and Hermite divided the new domain which opened before them. Their works are so interlaced in this fraternal rivalry that it would be difficult and hardly desirable to distinguish with precision the part done by each in their common work. It seems, however, that we can attribute in particular to M. Hermite the law of reciprocity, the discovery of associated covariants, that of skew invariants, and the formation of the complete system of covariants of cubic and biquadratic forms and of invariants of the form of the fifth order.

These important researches in arithmetic and algebra did not satisfy his activity; he pursued at the same time his studies on transcendants. In a series of memorable investigations he solved the problem of the transformation of hyperelliptic functions, and from the developments in series of the elliptic functions he deduced important formulæ relative to the number of classes of quadratic forms.

He laid the foundations of the theory of modular functions at the same time and solved in its details the difficult question of their transformation, thus giving far in advance a model for those who have elaborated and generalized this theory in our day.

The impression produced on geometers by the ensemble of these works is reflected very well in a picturesque phrase which we received at that time from the lips of Lamé: "To read the memoirs of M. Hermite makes one's flesh creep."

In 1856, at the age of thirty-four years, M. Hermite entered the Institute; in 1862 a chair was created for him at the Ecole Normale; a little later he became professor also at the Ecole Polytechnique and at the Sorbonne.

At this epoch the course of instruction, it must be said, was hardly up to date. The great discoveries by which Gauss, Abel, Jacobi, and Cauchy had transformed the science during half a century were passed over in silence, as if they were of interest only to the initiated few. M. Hermite cast them boldly into the public domain. This happy audacity has borne its fruits. Witness our young and brilliant school of geometers; they were all students of Hermite, and owe a great deal of their success to his lectures and kindly encouragement.

His peaceful realm was not confined to our borders. M. Hermite kept up correspondence throughout learned Europe, and young talent anywhere could count on his counsel and support. Neither the duties of instruction nor even the afflictions of age could do injury to the fertility of his mind. In fact, from this second period date a great number of beautiful works which yield precedence in no way to the works of his youth. However, an appreciable evolution took place in the object of his investigations. Arithmetic and algebra which had predominated up to this time gave way to the integral calculus

The transition is made by a celebrated memoir on the equation of the fifth degree, whose solution he gives by means of elliptic functions. Then follow the researches on interpolation, on new methods of developing functions in series of polynomials, on the discontinuities of definite integrals which depend on a parameter, etc.

In the theory of elliptic functions M. Hermite discovered a fundamental formula which permits their decomposition into simple elements, and consequently their integration. He was the first to study doubly periodic functions of the second species.

We arrive finally at the memoir on the exponential function, worthily crowning his long researches on the developments in continued fractions. He made clear that the number e is transcendental. Lindemann has since established that the number π is also transcendental. The solution of the problem of the quadrature of the circle, so vainly sought throughout all the centuries, is therefore demonstrated to be impossible.

We can legitimately claim for M. Hermite a share in this beautiful result, because it has been attained by following the process which he employed for the exponential. But one would give a very incomplete idea of the rôle of great minds by measuring them exclusively by the new truths which they have explicitly enunciated. The methods which they have bequeathed to their successors, leaving to their care the application of these methods to new problems perhaps unanticipated by themselves, constitute another part, sometimes the principal part, of their glory, as the example of Leibnitz shows.

For almost a century we have labored to develop the fertile germs which Gauss and Cauchy have sown in their writings ; it will be the same with Hermite. Behold two examples to prove it :

The remarkable group of substitutions which he encountered in his researches on the transformation of abelian functions serves as the essential element for the solution of a problem altogether different, the resolution of equations by radicals. It appears again in the discussion of the second variation of definite integrals.

The quadratic forms having conjugate variables are the indispensable fundament of investigations in the reduction of the most general forms with real and complex coefficients.

M. Hermite loved science for its own sake and was but little occupied with its applications ; these came spontaneously and by superaddition. To Lamé's equation, whose integration constitutes the last of his great works, he attached quite a series of problems in mechanics: rotation of a solid, determination of the elastic curve, oscillations of the conical pendulum.

To form a correct idea of the place occupied by M. Hermite in the mathematical world, one should have taken part with us in the memorable fêtes of his jubilee in 1892. All his friends, his disciples, his admirers shared in this impressive ceremony ; all the learned societies of Europe sent either addresses or delegates.

The same year saw the jubilee of Pasteur. To-day Pasteur and Hermite are no more ; there remains for us only the souvenir of their examples and their works, but these are sufficient to immortalize their memory.

Permit us in concluding to express a wish in behalf of the section of geometry. The work of Hermite is very scattered; in addition to the principal memoirs, it contains many letters and short notes disposed here and there ; but all bear la griffe du lion. The Academy would honor itself and render a great service to geometers by undertaking the publication of the complete works of Charles Hermite.

NOTES.

THE committee in charge of the colloquium to be held in August in connection with the summer meeting of the AMERICAN MATHEMATICAL SOCIETY is now able to announce that Professor OSKAR BOLZA has consented to give a course of lectures on "The calculus of variations, in particular Weierstrass's discoveries."

THE ninth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago, on Saturday, April 6, 1901, the first session opening at 10 o'clock A. M., in the Ryerson Physical Laboratory. The Christmas meeting will be held at Northwestern University, Evanston, Ill., on Friday and Saturday, December 27 and 28. Titles, abstracts, and time requirements of papers to be read at the April meeting should be in the hands of the Secretary of the Section, for the use of the programme committee, not later than March 20.

THE January number (volume 23, number 1) of the *American Journal of Mathematics* contains the following papers: "Die Typen der linearen Complexe rationaler Curven in R_r," by S. KANTOR ; "Transformations of systems of linear differential equations," by E. J. WILCZYNSKI ; "Distribution of the ternary linear homogeneous substitutions in a Galois field into complete sets of conjugate substitutions," by L. E. DICKSON ; "Distribution of the quaternary linear homogeneous substitutions in a Galois field into complete sets of conjugate substitutions," by T. M. Putnam; "On the determination and solution of the metacyclic quintic equation with rational coefficients," by J. C. Glashan; "Construction of the geometry of euclidean n-dimensional space by the theory of continuous groups," by E. O. Lovett; "A table of class numbers for cubic number fields," by L. W. Reid; "On certain properties of the plane cubic curve in relation to the circular points at infinity," by R. A. Roberts. The number also contains portraits of Professors George Salmon and Mittag-Leffler.

The following papers were read at a meeting of the London mathematical society held on January 10, 1901: "On streaming motions past cylindrical boundaries," by Professor A. E. H. Love; "On some cases of the solution of $x^{p^n-1} \equiv 1$, mod. p," by Professor F. S. Carey; "On the zeros of Bessel's functions," by Mr. E. W. Barnes; "A proof of the third fundamental theorem in Lie's theory of continuous groups," by Mr. J. E. Campbell.

At a meeting of the Edinburgh mathematical society held on December 14, 1900, Dr. J. A. Third read a paper on triangles triply in perspective, and Professor R. E. Allardice communicated a paper on four circles touching a common circle. At the meeting of the same society held on January 11, 1901, Professor Allardice read a paper on the nine point conic.

Sylvester Prize. The first award of the Sylvester Prizes of the Johns Hopkins University (see the present volume of the Bulletin, page 191) was made on February twenty-second. The prize is a handsome bronze medallion of the late Professor Sylvester, framed in oak. In the course of the ceremonies, President Gilman announced the award as follows: "The first impression of this tablet is presented to Lord Kelvin, who lectured here on 'The nature of light' in 1884. This will be offered to Lord Kelvin, at the time of the jubilee of the University of Glasgow in June next, by a delegate from this university. The second copy of the tablet is now offered to Professor Simon Newcomb, a distinguished astronomer, who has been a friend of the university from its inception, and who guided the affairs of the mathematical department for many years."

The Prussian academy of sciences has conferred the Helmholtz medal upon Sir George Gabriel Stokes. The medal has hitherto been conferred only on Professor VIRCHOW and LORD KELVIN.

THE well known treatise on analysis of Professor E. PICARD is to appear in a new edition, of which the first part of the first volume has just come from the press of M. Gauthier-Villars, of Paris.

PROFESSOR GINO LORIA has issued a reprint of an academical address dealing with the history of mathematics which he recently presented to the University of Genoa, under the title "The transfigurations of a science."

IT is announced that Professor A. MAYER is to retire from the editorial board of the *Mathematische Annalen*, and that Professor D. HILBERT is to take his place. The board will thus consist of Professors F. KLEIN, W. DYCK, and D. HILBERT.

UNIVERSITY OF MUNICH.—The following courses in mathematics are offered during the summer semester, 1901: By Professor G. BAUER: Theory of algebraic plane curves; Mathematical seminar.—By Professor F. LINDEMANN: Analytical geometry of space; Definite integrals and Fourier's series; Mathematical seminar.—By Professor H. SEELIGER: celestial mechanics, second course, methods of Jacobi and Hamilton; Exercises with the instruments of the observatory.—By Professor A. PRINGSHEIM: subjects to be announced later.—By Dr. K. DÖHLEMANN: Descriptive geometry, second course, with exercises; The imaginary in geometry.—By Dr. E. VON WEBER: Encyclopædia of elementary geometry; Modern geometric theories.—By Dr. K. SCHWARZSCHILD: Methods of interpolation and mechanical integration.

PROFESSOR M. CANTOR, of Heidelberg, has been elected a correspondent of the St. Petersburg academy of sciences.

PROFESSOR H. S. WHITE, of Northwestern University, has received leave of absence and will remain abroad until October.

THE death is announced of Professor M. F. KOVALSKIJ, professor of mathematics at Charkow.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CANDY (A. L.). The elements of analytic geometry. Lincoln, Neb., published by the author, 1900. 8vo. 303 pp. Cloth. $2.00

FORMULAIRE de mathématiques, publié par G. Peano. Edition de l'an 1901 (tome III de l'édition complète). Turin, Bocca, 1901. 8vo. 8 + 231 pp.

HAYES (E.). Calculus, with applications; an introduction to the mathematical treatment of science. Boston, Allyn & Bacon, 1900. 12mo. 7 + 162 pp. Cloth. $1.20

HILBERT (D.). Les principes fondamentaux de la géométrie Traduit par L. Laugel. Paris, Gauthier-Villars, 1900. 4to. 114 pp.

LAUGEL (L.). See HILBERT (D.).

LEJEUNE-DIRICHLET (P. G.), Die Darstellung ganz willkürlicher Funktionen durch Sinus- und Cosinus-Reihen (1837), und SEIDEL (P. L.), Note uber eine Eigenschaft der Reihen welche diskontinuierliche Funktionen darstellen (1847). Herausgegeben von H. Liebmann, Leipzig, Engelmann, 1900. 12mo. 58 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 116.) Cloth. M. 1.00

LIEBMANN (H.). See LEJEUNE-DIRICHLET (P. G.).

PEANO (G.). See FORMULAIRE.

PIERPONT (J.). Galois's theory of algebraic equations. Six lectures delivered at the Buffalo Colloquium in September, 1896. (Published in the *Annals of Mathematics*. ser. 2, Vols. 1 and 2, 1900.) Cambridge, Harvard University, 1900. 4to. 67 pp. $0.75

PUZYNA (I.). Teorya funkcji analitycznych (Theory of analytic functions). Part 2. Lemberg, 1900. 8vo. 16 + 693 pp. M. 16.00

REID (L. W.). Tafel der Klassenanzahlen für kubische Zahlkörper. (Diss.) Göttingen, 1899. 8vo. 75 pp

SAILER (E.). Die Aufgaben aus der Differential- und Integralrechnung und aus der analytischen Geometrie, welche bei der Prufung für das Lehramt der Mathematik und Physik an den k. bayerischen humanistischen und technischen Unterrichtsanstalten in den Jahren 1873 bis 1893 gestellt wurden. München, Ackermann, 1900. 8vo. 187 pp. M. 4.80

SEIDEL (P. L.). See LEJEUNE-DIRICHLET.

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L RARY

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 23, 1901. Thirty-two persons were in attendance at the two sessions, including the following twenty-eight members of the Society:

Professor Anne L. Bosworth, Professor Joseph Bowden, Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Professor T. S. Fiske, Mr. A. S. Gale, Mr. F. A. Giffin, Mr. Edwin Haviland, Jr., Dr. H. E. Hawkes, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. G. H. Ling, Dr. James Maclay, Dr G. A. Miller, Mr. H. B. Mitchell, Professor E. H. Moore, Professor W. F. Osgood, Professor James Pierpont, Dr. M B. Porter, Professor P. F. Smith, Professor Henry Taber, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor Eliakim Hastings Moore, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor John F. Downey, University of Minnesota, Minneapolis, Minn.; Professor Frederick C. Ferry, Williams College, Williamstown, Mass ; Mr. Henry T. Gerrans, Oxford University, Oxford, England ; Mr. Edwin Haviland, Jr., New York, N. Y.; Professor Augustus E. H. Love, Oxford University, Oxford, England ; Mr. V. R. Thyagarajaiyar, Bangalore, India. Two applications for membership were received. Action was taken by the Council toward making the Society's library accessible for the use of the members.

The following papers were read at this meeting:

(1) Dr. H. E. HAWKES: "Note on Hamilton's determination of irrational numbers."

(2) Professor E. B. VAN VLECK: "On the convergence of continued fractions with complex elements."

(3) Dr. M. B. PORTER: "On linear homogeneous finite difference equations, with applications to certain theorems of Sturm."

(4) Professor L. E. DICKSON: "Concerning real and complex continuous groups."

(5) Professor E. O. LOVETT: "An application of infinite groups to non-euclidean geometry."

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(6) Professor E. O. LOVETT: "Contact transformations which change asymptotic lines into lines of curvature."

(7) Professor H. B. NEWSON: "Indirect circular transformations and mixed groups."

(8) Mr. W. B. FITE: "On metabelian groups that cannot be groups of cogredient isomorphisms."

(9) Dr. EDWARD KASNER: "On algebraic potential curves."

(10) Professor MAXIME BÔCHER: "Green's functions in space of one dimension."

(11) Dr. H. E. HAWKES: "Estimate of Benjamin Peirce's linear associative algebra."

(12) Dr. G. A. MILLER: "On holomorphisms and primitive roots."

(13) Dr. EDWARD KASNER: "Theorems on collinear lines in space."

(14) Mr. C. W. M. BLACK: "Decomposition of a form in n variables in an arbitrary domain with respect to a prime ideal modulus."

(15) Professor MAXIME BÔCHER: "An elementary proof of a theorem of Sturm."

(16) Dr. L. P. EISENHART: "Surfaces whose first and second fundamental forms are the second and first respectively of another surface."

(17) Dr. L. P. EISENHART: "Possible triply asymptotic systems of surfaces."

(18) Dr. H. F. STECKER: "On the determination of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by algebraic curves."

(19) Professor MAXIME BÔCHER: "Non-oscillatory linear differential equations of the second order."

Mr. Fite was introduced by Dr. G. A. Miller, and Mr. Black by Professor James Pierpont. In the absence of the authors, Professor Bôcher's first paper was read by Professor W. F. Osgood, and the papers of Professor Dickson, Professor Lovett, Professor Newson, Dr. Eisenhart, Dr. Stecker, and the second and third papers of Professor Bôcher were read by title.

Professor Newson's paper appeared in the March number of the BULLETIN. The present number contains the first paper of Professor Bôcher, the first paper of Dr. Hawkes, and the second paper of Dr. Eisenhart. The papers of Professor Dickson and Dr. Miller, and Professor Bôcher's second paper will appear in later numbers. The paper of Dr. Stecker and the third paper of Professor Bôcher will appear in the *Transactions*. Abstracts of the other papers are given below.

The paper of Professor E. B. Van Vleck, which will be published in the *Transactions*, gives first a résumé of the few existent theorems of a general character upon the convergence of continued fractions with complex elements. It then goes on to develop a set of equations which appears to be both new and fundamental for the study of continued fractions with complex elements. These equations are applied to the derivation of certain theorems, the first of which is as follows: If in a continued fraction

$$\frac{1}{\alpha_1 + \iota\beta_1 +} \frac{1}{\alpha_2 + \iota\beta_2 +} \frac{1}{\alpha_3 + \iota\beta_3 +} \cdots$$

the β-constituents with even subscripts have one sign and those with odd subscripts the opposite sign, while the α-constituents have a common sign, the continued fraction will converge if $\sum_{n=1}^{\infty} \left| \alpha_n + \iota\beta_n \right|$ is divergent. On the other hand, if $\sum | \alpha_n + \iota\beta_n |$ is convergent, the even convergents approach one limit and the odd convergents another. It is further shown that neither the numerator nor the denominator of any one of the convergents of the continued fraction can vanish. This also holds true for the convergents of any continued fraction in which either the α_n or the β_n meet the conditions of the first theorem.

Continued fractions are next considered in which a finite number of the α-constituents or of the β-constituents (but not of both) fail to fulfill the conditions imposed in the first theorem, and such continued fractions are shown to converge. Under certain restrictions, also, an infinite number of exceptional elements may be admitted. The criterion for convergence stated in the first theorem accordingly differs from criteria hitherto obtained in that it admits of certain irregularities in the continued fraction whose positions therein are entirely arbitrary.

In conclusion, the results of the paper are applied to algebraic continued fractions. Thus it appears that if each element λ_n of a continued fraction

$$\frac{1}{\lambda_1 +} \frac{1}{\lambda_2 +} \frac{1}{\lambda_3 +} \cdots$$

is a positive number or such a number multiplied into z, the zeros of the numerators and denominators of the convergents will all lie upon the negative half of the z-axis. If, furthermore, no element with even (respectively with odd) subscript contains z and if $\sum|\lambda_n|$ is divergent, the continued fraction will converge over at least three-quarters of the plane; namely, over the half plane in which x is positive and in those sections in which x is negative and $|y| > |x|$. Within this region the continued fraction represents an analytic function and never vanishes. On the other hand, if $\sum|\lambda_n|$ is convergent, the even and odd convergents approach different limits, and these limits are likewise analytic and never vanish within the same region.

The theorems given by Sturm in his first important memoir in *Liouville's Journal*, Volume I., were first arrived at, so the author states at the end of the paper, as limiting cases of certain analogous theorems concerning linear difference equations of the second order. Sturm's work never having been published, Dr. Porter's paper aims at its restoration.

In the sixteenth volume of the *Acta Mathematica*, M. Zorawski followed a suggestion of the late Sophus Lie and determined the deformation invariants of surfaces in ordinary space by the method of the theory of continuous groups. It is the object of Professor Lovett's first paper to employ the same method to design the mechanism for the construction of the deformation invariants of varieties in a point space of any number of dimensions some power of whose lineal element is a homogeneous function of the differentials of the point coördinates of the space, and in particular to point out how the known and possible deformation invariants of surfaces in ordinary space and the theorems relative thereto may be extended immediately to surfaces in any quadratic space, euclidean or non-euclidean, of three dimensions. This particular result for varieties having three dimensions is especially interesting since it constitutes a contribution from the theory of groups to the geometry of all such varieties, and this in the face of the fact that the theory of finite groups is limited in its application to the construction of the geometries of three-dimensional manifoldnesses as is evidenced by the recent work of Bianchi (*Memoirs of the Italian Society of Sciences*, 3d series, vol. 11) and Cotton (thesis presented to the Faculty of Sciences of Paris), who have found all such varieties whose lineal elements admit of continuous groups.

The conclusion drawn follows from the fact that the deformation invariants of any particular class are determined once for all for every variety of three dimensions by the integration of one and the same formal system of linear partial differential equations. To construct this system of linear equations it is necessary to effect an m-fold extension of the infinitesimal transformations of an infinite group. Such an extension can be proposed in an infinite number of ways. The latter problem has also been studied by Zorawski (*Proceedings of the Cracow Academy of Sciences*, 2d series, vol. 4), who extends the transformation relative to the derivatives of its own functions; and by Levi-Civita (*Transactions of the Venetian Academy of Sciences, Letters and Arts*, 7th series, vol. 5), who extends the transformation relative to the elements of any covariant and contravariant systems whatever. The present note extends the transformation in three different ways, namely relative to all partial derivatives of any function whose variation is known, with regard to all derivatives of any one variable considered as a function of the other point coördinates, and finally with reference to the derivatives of a system of any number of invariant functions. The corresponding systems of differential equations give generalizations of the so-called deformation invariants of Gauss, Beltrami, and Minding.

The geometry of contact transformations calls for the determination of those transformations which change a surface into another surface in such a manner that a remarkable family of curves on one surface is transformed into a remarkable family of curves on the other surface. Such, for example, are those establishing correspondences between asymptotic lines, lines of curvature, Darboux lines, geodesic circles, etc. Those transforming asymptotic lines into such are known to be either projective or dualistic. This result is generalized in a subsequent note. Dilatations and inversions are known to change lines of curvature into lines of curvature. These theorems relative to asymptotic lines and lines of curvature are to be found in Darboux's Theory of Surfaces, volume I. It is proposed in Professor Lovett's second paper to determine those transformations which change asymptotic lines into lines of curvature. One such transformation was found by Sophus Lie, namely his celebrated correspondence between straight lines and spheres, in his well known memoir in the fifth volume of the *Mathematische Annalen*.

It appears that the functions X, Y, Z, P, Q, capable of defining a contact transformation possessing the desired property, must satisfy a certain system of equations of condition. The problem of determining the forms of the functions X, Y, Z, P, Q, from these general equations of condition offers serious difficulties. However, the means are at hand for the complete resolution of the problem, for Lie has remarked that all contact transformations which change asymptotic lines into lines of curvature also transform straight lines into spheres, and the latter transformations have been found by the author in a paper soon to be published. Accordingly, to find all contact transformations which change asymptotic lines into lines of curvature it is only necessary to subject the types of line sphere transformations already found to the equations of condition, a process simple but extremely laborious. It can be verified, however, without much difficulty that the functions X, Y, Z, P, Q determined by the sets of equations

$$\omega_1 = \sum_1^4 (a_{1j} X + i a_{1j} Y + a_{2j} Z + a_{3j})\, x_j = 0,$$

$$\omega_1 = \sum_1^4 (a_{2j} X - i a_{2j} Y - a_{1j} Z + a_{4j})\, x_j = 0,$$

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = 1,\ i = \sqrt{-1},$$

$$\omega_1^x / \omega_2^x = \omega_1^y / \omega_2^y = \omega_1^X / \omega_2^X = \omega_1^Y / \omega_2^Y$$

$$\omega_3 = \sum_1^4 a_{1j} X - i a_{1j} Y + a_{2j} Z + a_{3j}) x_j = 0,$$

$$\omega_4 = \sum_1^4 (a_{2j} X + i a_{2j} Y - a_{1j}\, Y + a_{4j}) x_j = 0,$$

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = 1, \quad i = \sqrt{-1},$$

$$\omega_3^x / \omega_4^x = \omega_3^y / \omega_4^y = \omega_3^X / \omega_4^X = \omega_3^Y / \omega_4^Y$$

constitute two families of ∞^{15} transformations which change asymptotic lines of a surface into the lines of curvature of the transformed surface. The celebrated transformation of Lie is included in both families; we find it in the first by making

$$a_{21} = a_{42} = a_{33} = a_{14} = 1$$

and all the other constants zeros in the first set of equations, and also by putting

$$a_{11} = a_{32} = - a_{43} = - a_{24} = 1$$

and all the other coefficients zero in the second set of equations.

The above equations of condition can be simplified in several particulars. They reduce to fifteen in number if

$$B_1 = B_2 = B_3 = C_1 = C_2 = C_3 = D_1 = D_2 = D_3 = 0,$$

and these last equations are readily found to be compatible. They are satisfied by Lie's transformation; moreover for the latter we have also

$$\begin{vmatrix} X^x X^y \\ Y^x Y^y \end{vmatrix} = \begin{vmatrix} X_p X_q \\ Y_p Y_q \end{vmatrix} = 0, \quad A_2^2 - 4A_3A_1 = 0, \quad E_2^2 - 4E_3E_1 = 0.$$

It appears incidentally that the determinant

$$\begin{vmatrix} X^y X_p & X^y Y_p + X_p Y^y & Y^y Y_p \\ X^x X_p - X^y X_q & X^x Y_p + X_p Y^x - X^y Y_q - X_q Y^y & Y^x Y_p - Y^y Y_q \\ X^x X_q & X^x Y_q + X_q Y^x & Y^x Y_q \end{vmatrix}.$$

is zero for any contact transformation.

Mr. Fite shows 1° that a metabelian group of odd order which has a set of generators such that the order of one of them is not a divisor of the least common multiple of the orders of all the others cannot be the group of cogredient isomorphisms of any group; 2° that if a metabelian group of order p^α, where p is a prime, has a cyclical commutator subgroup and a group of cogredient isomorphisms with more than two independent generators, it can not be the group of cogredient isomorphisms of any group.

The curves considered in Dr. Kasner's first paper present themselves in the theories of functions, of equations, and of the potential. They are defined analytically by the vanishing of a rational integral solution of Laplace's equation $\varphi_{xx} + \varphi_{yy} = 0$; or, what is equivalent, by the vanishing of the real part of a rational integral function of $x + iy$. The paper gives geometric definitions of these potential curves, and some of their metric and projective properties. A potential curve is apolar to the degenerate class conic composed of the circular points at infinity; all of its polar conics are rectangular hyperbolas. Any curve of the nth order passing through the n^2 foci of a curve of the nth class is a potential curve; conversely, any potential curve passes through the foci of an infinite number of systems of confocal curves of the nth class. Conjugate potential curves intersect orthogonally in a set of such foci. This last result is applied to the theory of equations.

In 1870 Benjamin Peirce enumerated all systems of hypercomplex numbers of certain types in less than seven units. His principles of classification are five in number and are shown to be identical or closely related to the principles of classification suggested by the theory of continuous groups and adopted by Scheffers (*Mathematische Annalen,* vol 39 (1891), p. 253). In Dr Hawkes's second paper various criticisms of Peirce's work by Cayley, Study, and others are noted and discussed.

Dr. Kasner's second paper is in abstract as follows: Consider any five collinear lines L_1,L_2,L_3,L_4,L_5, *i. e* , five lines having a common tractor M_0. In addition to M_0, any four of these lines as L_2,L_3,L_4,L_5 have another tractor M_1. It is proved in the first place that the five lines $M_1,M_2.M_3,M_4.M_5$, thus derived, are themselves collinear. Denote their common tractor by L_0. The remainder of the paper is devoted to the study of the configuration composed of the twelve lines L and M and of the thirty points and thirty planes which they determine. The same configuration is determined by any five of the lines L, or any five of the lines M. The anharmonic ratios of the various points and planes are connected by simple relations; all are expressible in terms of four, the fundamental absolute invariants. Finally these relations are employed to prove that the derived set M_1,M_2,M_3,M_4,M_5 is correlative to the original set L_1,L_2,L_3,L_4,L_5.

In Mr. Black's paper it was shown first that all coefficients of any possible factor can be brought within the residual system of the modulus, and then the irreducible factors can be determined by a finite number of trials. In the second part of the paper, the uniqueness of decomposition is shown, first for a single variable, and then by induction for n variables, the main part of the discussion being directed to bringing the problem into a form such that reasoning similar to that used for the proof of the uniqueness of algebraic decomposition may be applied.

Dr. Eisenhart's first paper is in abstract as follows: In view of the fact that there is a unique surface corresponding to two quadratic differential forms, whose coefficients satisfy the Codazzi and Gauss equations, the question is asked whether it is ever possible for the first and second fundamental forms of a surface to be the second and first respectively of a second surface. In answer it is found that there is a class of surfaces possessing this property; that they are ruled surfaces with imaginary generatrices; that the sphere of radius unity is the only real surface of the class; and, furthermore, that the second surface differs from the first only by a translation in space.

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GREEN'S FUNCTIONS IN SPACE OF ONE DIMENSION.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, February 23, 1901.)

I WISH to make a brief communication to the Society of some results which I have obtained, reserving proofs and further developments for another occasion.

By a Green's function is ordinarily understood a solution of Laplace's equation which vanishes on the boundary of a certain region, and within this region is discontinuous at only one point, where it becomes infinite like $1/r$ or $\log r$, according as we are dealing with Laplace's equation in three or in two dimensions. We may speak of this as a Green's function of the first kind, in distinction to the generalized Green's function for which more complicated conditions than the mere vanishing of the function are imposed on the boundary of the region.

This fundamental conception has been generalized by replacing Laplace's equation on the one hand by other homogeneous linear partial differential equations of the second order (cf. Encyklopädie, volume 2, p. 516); on the other hand by the ordinary differential equation $d^2y/dx^2 = 0$, which may be called Laplace's equation in space of one dimension (cf. Burkhardt, *Bulletin de la Société mathématique de France*, volume 22 (1894), p. 71). This suggests at once the possibility of considering in place of Laplace's equation the general ordinary homogeneous linear differential equation of the second order. I find that we can not only do this, but can go further by considering at once linear differential equations of the nth order

$$(1) \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 \qquad (n \geqq 2).$$

We will assume here, in order to avoid all complications, that $p_1, p_2, \cdots, p_n$ are real functions of the real variable x which are continuous at every point of the interval

$$(J) \qquad a \leqq x \leqq b.$$

Let ξ be a point between a and b, and let k be a positive integer less than n $(1 \leqq k \leqq n-1)$.

We seek now a function $G(x)$ having the following properties:

(a) At every point of (J), except ξ, the function G is continuous, has continuous derivatives of the first n orders, and satisfies (1).

(b) At the point a, G and its first $n-1$ derivatives satisfy k linearly independent, but otherwise arbitrarily chosen, homogeneous linear relations.

(c) At the point b, G and its first $n-1$ derivatives satisfy $n-k$ linearly independent, but otherwise arbitrarily chosen, homogeneous linear relations.

(d) At ξ, G and its first $n-2$ derivatives are continuous.

(e) At ξ, G has a derivative D_+ of order $n-1$ taken in the positive direction, and a derivative D_- of order $n-1$ taken in the negative direction, and

$$D_+ - D_- = -1.$$

It turns out that in general there exists one and only one function satisfying these conditions. This function we call a Green's function, and denote it by $G(x, \xi)$. The effect of interchanging x and ξ is to change (1) into the adjoint equation (cf. Schlesinger's Handbuch, Vol. I., p. 53) and in general to alter the relations (b) and (c); otherwise conditions (a) — (e) remain true.

I will mention here only one application of the Green's function thus defined:

The solution of the non-homogeneous equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = p,$$

which satisfies at a and b the same conditions which are satisfied there by G, is given by the formula

$$y = -\int_a^b G(x, \xi)\, p(\xi)\, d\xi.$$

A special case deserves mention on account of its simplicity, viz., the case where conditions (b) reduced to

$$G(a, \xi) = G'(a, \xi) = \cdots = G^{[k-1]}(a, \xi) = 0$$

and conditions (c) to

$$G(b, \xi) = G'(b, \xi) = \cdots = G^{[n-k-1]}(b, \xi) = 0,$$

accents denoting differentiation with regard to x. One of the simplifications introduced by this specialization is that when x and ξ are now interchanged, the effect is to interchange a and b in conditions (b) and (c). It will be seen that for a given equation (1) and a given interval (J) there are $n - 1$ Green's functions of this special sort, obtained by giving to k the values $1, 2, \cdots, n - 1$. These functions might properly be called Green's functions of the first kind, since when $n = 2$ there is only one of them, namely the Green's function, which vanishes at both ends of (J).

GÖTTINGEN, GERMANY,
February 8, 1901.

ON A SYSTEM OF PLANE CURVES HAVING FACTORABLE PARALLELS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1900.)

THE type of scrolls contained in a linear congruence, and having factorable asymptotic lines, gives rise to a class of plane curves whose parallels have a similar property.

It has been shown * that the spherical images of such scrolls have factorable lines of curvature and a method was given by means of which a scroll of this type could be transformed into an annular surface having only plane lines of curvature, the curve of each section breaking up into two factors. The planes of the system all belong to the same axial pencil, and the locus of centers of the generating spheres lies in the bisecting plane of the angle between the two plane directors.

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* V. Snyder, "On a special form of annular surface." *Amer. Jour. of Math.*, vol. 23.

When the axis of this pencil of planes lies in the plane at infinity, the director planes become antiparallel, the generating spheres have a constant radius, and their envelope is a tubular surface. The procedure is as follows : take any two parallel lines

$$(q)\quad y=\kappa t,\ z=0\,;\qquad (p)\quad y+\kappa t=0,\ z=0\,;$$

these lines go over into two parallel planes in sphere space. Take the conjugate of (p) with regard to the complex c, image of point spheres. This line (p') is $x=\kappa z$, $t=0$.

The image of this line (which does not cut (q)) is the same plane as that of p, but now regarded as generated by its second pencil of minimum lines, so that the direction of the normal is reversed. Any scroll having these two lines (p') and (q) for directrices will be defined by

$$\frac{y-\kappa t}{z}=\lambda,\quad \frac{x-\kappa z}{t}=\mu,$$

the relation between λ, μ being determined by the third director curve $f(\lambda,\mu)=0$.

The spherical image of this scroll has for its equation

$$(1)\qquad \begin{vmatrix} \tau & 0 & \zeta & -(\xi+i\eta) \\ 0 & \tau & -(\xi-i\eta) & -\zeta \\ 0 & 1 & -\lambda & -\kappa \\ 1 & 0 & -\kappa & -\mu \end{vmatrix}=0$$

or $$\left(\xi-\frac{(\mu+\lambda)\tau}{2}\right)^2+\left(\eta-\frac{(\lambda-\mu)i\tau}{2}\right)^2+\zeta^2=\kappa^2\tau^2,$$

i. e., it is the envelope of a sphere of constant radius κ, whose center moves in the plane $\zeta=0$, subject to the law $f(\lambda,\mu)=0$. In particular, if $f(\lambda,\mu)=0$ defines a scroll of the type mentioned, the lines of curvature of the second system belonging to this tube will be factorable. But it is at once evident that these curves are simply the parallels of the locus of centers.

Conversely, if such a tube is generated from a curve whose parallels are known to be factorable, the general annular surface derived from it and also the scroll, its line space image, will have the same property. Starting from this space, let $\varphi(\alpha,\beta)=0$ be the equation of any plane curve ; then

$$(\xi-(m\beta+\alpha))^2+\eta^2+\zeta^2=\beta^2+m^2\beta^2,\quad m=\frac{d\beta}{d\alpha},$$

is the equation of a sphere whose envelope is the surface of revolution formed by rotating $\varphi(\alpha, \beta) \equiv \varphi(\xi, \eta) = 0$ about the ξ-axis. This surface is an annular one contained in a linear congruence, its two director planes being $\zeta \pm i\eta = 0$. The image in line space of this sphere is

$$\beta\sqrt{1+m^2}z = x - (m\beta + \alpha)t,$$
$$(m\beta + \alpha)z = y - \beta\sqrt{1+m^2}t\,;$$

and the images of the directors are as follows:

$$\zeta + i\eta = 0 \text{ becomes } z + t = 0,\ x + y = 0,$$
$$\zeta - i\eta = 0 \quad \text{``} \quad z = t, \quad x = y.$$

The generators of the scroll having these two lines for directrices must be of the form

$$\frac{x+y}{z+t} = \lambda, \quad \frac{x-y}{z-t} = \lambda\,;$$

making them identical with the above form, we have

$$\frac{\mu+\lambda}{2}z = x + \frac{(\lambda-\mu)}{2}t, \quad \frac{(\mu-\lambda)}{2}z = y - \frac{(\lambda+\mu)}{2}t\,;$$

from which

$$\frac{\mu+\lambda}{2} = \beta\sqrt{1+m^2}, \quad \frac{\mu-\lambda}{2} = m\beta + \alpha.$$

But when the directrices are transformed, the meaning of λ, μ remains the same and equation (1) shows what these directrices give when so transformed that the spherical image shall be a tubular surface. Hence: *the equations*

$$x = \beta\sqrt{1 + \left(\frac{d\beta}{d\alpha}\right)^2}, \quad y = i\left(\alpha + \beta\frac{d\beta}{d\alpha}\right)$$

$$\varphi(\alpha, \beta) = 0, \quad \varphi(\alpha, \beta) \neq \varphi(\alpha, -\beta)$$

define a curve whose parallel curves break up into two distinct factors, corresponding to the positive and negative directions of the normal, i. e., toward or away from the center of curvature from the point on the curve.

It is not difficult to generate other surfaces, all of whose lines of curvature of both systems are plane, and those of one system factorable; any "surface moulure" having a factorable director curve is of this type,* but the above surfaces are the only annular surfaces of the class. The second mantle of the surface of centers is the cylinder erected on the evolute of the locus of centers of the generating spheres. When this cylinder becomes a straight line the surface is a torus.

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* See Enneper, "Untersuchungen uber die Flachen mit planen und sphärischen Krummungslinien," *Göttinger Abhandl.*, vol. 23, cf. p. 45, eq. (68); and Bianchi, Differentialgeometrie, ch. 5, § 74.

The factorability of the asymptotic lines of a scroll contained in a linear congruence depends entirely upon the configuration at the pinch points, all of which are situated on the linear directrices. When such a scroll is transformed into a tubular surface, these points go into the circular points of the plane containing the locus of centers; hence the configuration of the curve at the circular points determines whether its parallels be factorable or not.

In particular, when the equation of the scroll is of the form

$$\mu\varphi(\lambda)+f(\lambda)=0,$$

the asymptotic lines are factorable when

$$\begin{vmatrix}\varphi & f\\ \varphi' & f'\end{vmatrix}\neq 0$$

is a perfect square (or a constant); but this is the condition that all of the pinch points are double, so that the n generators from each point of the multiple directrix are always the same, *i. e.*, κ real, $n-\kappa$ imaginary, κ being constant.

Scrolls of this type exist for every value of n.

The corresponding condition for the curve is that $\alpha^2+\beta^2$ should be equal to a square (or a constant), $\alpha^2+\beta^2$ being the tangential equation of the circular points. For curves of class 2 or 4 this expresses that the foci of its dual curve must all be singular, but this is not true when the class is higher. Thus the dual of the general bicircular quartic has not factorable parallels, but those of the cartesian ovals, limaçon, cardioid, etc., have—the latter giving the cubic, $a(x^2+y^2)=(a+2x)^3$.

This equation $\varphi^2(\alpha,\beta,\gamma)(\alpha^2+\beta^2)=f^2(\alpha,\beta,\gamma)$ is of more general application than to the $(n, 1)$ scrolls and seems necessary and sufficient in every case.*

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* This equation is derived in a different way in Salmon's Higher plane curves, 3d edition, p. 103.

CORNELL UNIVERSITY,
October 30, 1900.

POSSIBLE TRIPLY ASYMPTOTIC SYSTEMS OF SURFACES.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, February 23, 1901.)

In my note, entitled "A demonstration of the impossibility of a triply asymptotic system of surfaces,"* which was read before the American Mathematical Society at its December meeting, I failed to take account of an exception which presents itself in the discussion. It was found that in order that there exist a triply asymptotic system of surfaces, it is necessary and sufficient that the double system of asymptotic lines of each surface shall at the same time be geodesic lines on the surfaces. This double condition can be satisfied only when the asymptotic lines in both systems are rectilinear generators of the surface. The quadric surfaces are known to possess this peculiar property, and for the hyperboloid of one sheet and the hyperbolic paraboloid these generatrices are real. Hence instead of the general negation previously given we have the qualified one:

The only triple systems of surfaces cutting mutually in the real asymptotic lines of these surfaces are composed of properly associated families of hyperboloids of one sheet and hyperbolic paraboloids.

One such system can be gotten as follows:

As, in the previous note, we consider space referred to any system of curvilinear coördinates ρ_1, ρ_2, ρ_3 and let the cartesian coördinates x, y, z of a point with respect to fixed rectangular axes be given in terms of ρ_1, ρ_2, ρ_3 by the equations

$$(1)\qquad x = f(\rho_1, \rho_2, \rho_3), \quad y = \varphi(\rho_1, \rho_2, \rho_3), \quad z = \psi(\rho_1, \rho_2, \rho_3).$$

We have remarked that the coefficients of the system

$$(2)\qquad \begin{aligned} \frac{\partial^2\theta}{\partial\rho_1^2} &= a_{11}\frac{\partial\theta}{\partial\rho_1} + a_{12}\frac{\partial\theta}{\partial\rho_2} + a_{13}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_2^2} &= a_{21}\frac{\partial\theta}{\partial\rho_1} + a_{22}\frac{\partial\theta}{\partial\rho_2} + a_{23}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_3^2} &= a_{31}\frac{\partial\theta}{\partial\rho_1} + a_{32}\frac{\partial\theta}{\partial\rho_2} + a_{33}\frac{\partial\theta}{\partial\rho_3}, \end{aligned}$$

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* Bulletin, January, 1901.

can be so determined that it will admit x, y, z, as particular simultaneous solutions. And we have found that the necessary and sufficient condition that the families of surfaces

$$\rho_1 = \text{const.}, \quad \rho_2 = \text{const.}, \quad \rho_3 = \text{const.},$$

shall cut one another in their asymptotic lines is expressed by

$$(3) \qquad a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.$$

Consider now in particular the case where x, y, z are such that they satisfy the equations

$$(4) \qquad \frac{\partial^2 \theta}{\partial \rho_1^2} = 0, \quad \frac{\partial^2 \theta}{\partial \rho_2^2} = 0, \quad \frac{\partial^2 \theta}{\partial \rho_3^2} = 0.$$

In order that this system may be consistent with system (2), it is necessary that, either all the coefficients in the right hand members of the latter be zero, or that x, y, z, as functions of ρ_1, ρ_2, ρ_3, satisfy the condition

$$\frac{\partial(x, y, z)}{\partial(\rho_1, \rho_2, \rho_3)} = 0,$$

i. e., that there exist a relation between x, y, z. As this is impossible, we must have

$$(5) \qquad a_{11} = a_{12} = \cdots = a_{33} = 0.$$

It is evident that the most general expressions which x, y, z can have under the conditions (4) are

$$(6) \qquad \begin{aligned} x &= a_1\rho_1\rho_2 + b_1\rho_1\rho_3 + c_1\rho_2\rho_3 + d_1\rho_1 + e_1\rho_2 + f_1\rho_3 + g_1, \\ y &= a_2\rho_1\rho_2 + b_2\rho_1\rho_3 + c_2\rho_2\rho_3 + d_2\rho_1 + e_2\rho_2 + f_2\rho_3 + g_2, \\ z &= a_3\rho_1\rho_2 + b_3\rho_1\rho_3 + c_3\rho_2\rho_3 + d_3\rho_1 + e_3\rho_2 + f_3\rho_3 + g_3, \end{aligned}$$

where a_1, b_1, $\cdots$, g_3 are arbitrary constants. The cartesian coördinates of a point on one of the surfaces $\rho_3 = \text{const.}$ are given as the following expressions of the parameters ρ_1 and ρ_2, which evidently refer to the double system of lines in which the surface is cut by the two families of surfaces $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$,

$$(7) \qquad \begin{aligned} x &= \alpha\rho_1\rho_2 + \beta\rho_1 + \gamma\rho_2 + \delta, \\ y &= \alpha_1\rho_1\rho_2 + \beta_1\rho_1 + \gamma_1\rho_2 + \delta_1, \\ z &= \alpha_2\rho_1\rho_2 + \beta_2\rho_1 + \gamma_2\rho_2 + \delta_2, \end{aligned}$$

where the expressions for α, $\cdots$, δ_2, as functions of ρ_3, a_1, $\cdots$, g_3, are readily found by comparing systems (7) and (6).

Eliminating ρ_1 and ρ_2 by means of Sylvester's dialytic method, we find the following equation of the second degree:

$$(8) \quad (\alpha_1\beta_2)(\alpha_1\gamma_2)x^2 + (\alpha\beta_2)(\alpha\gamma_2)y^2 + (\alpha\beta_1)(\alpha\gamma_1)z^2 - [(\alpha_1\beta_2)(\alpha\gamma_2)$$

$$+ (\alpha_1\gamma_2)(\alpha\beta_2)]xy + [(\alpha_1\beta_2)(\alpha\gamma_1) + (\alpha_1\gamma_2)(\alpha\beta_1)]xz -$$
$$[(\alpha\beta_2)(\alpha\gamma_1) + (\alpha\gamma_2)(\alpha\beta_1)]yz + Ax + By + Cz + D = 0,$$

where, for the sake of brevity, we have written

$$(\alpha_i\beta_k) \equiv \alpha_i\beta_k - \alpha_k\beta_i,$$

and A, B, C, D are functions of $\alpha, \cdots, \gamma_3$, whose form is of no consequence in the discussion except that they are finite for finite values of the latter.

Denoting by Δ the discriminant of the terms of second degree in (8), we find that

$$(9) \qquad \Delta = 0.$$

In consequence of this and of the character of A, B, C, as remarked, the quadric whose equation is (8) has its center at an infinite distance. By giving to ρ_3 successive values in this equation, we get the equations of the form

$$f(x, y, z) = \text{const.},$$

defining the family of surfaces $\rho_3 = \text{const.}$ From (6) and (7) we see that the asymptotic lines on these surfaces are real when ρ_3 and $a_1, \cdots, g_3$ are real; hence these surfaces $\rho_3 = \text{const.}$ are of total negative curvature. Moreover, these asymptotic lines are distinct for general values of $a_1, \cdots, g_3$. Hence $\rho_3 = \text{const.}$ defines a family of hyperbolic paraboloids.

Since the formulæ (6) are perfectly symmetrical with respect to ρ_1, ρ_2, ρ_3, it follows that in a similar manner we can show that $\rho_1 = \text{const.}$ and $\rho_2 = \text{const.}$ define respective families of hyperbolic paraboloids. Hence the formulæ (6) serve to define a triply asymptotic system of hyperbolic paraboloids.

Consider the surface $\rho_3 = k_3$, where k_3 is a fixed constant. The cartesian coördinates of a point on this surface are given by (7), in which ρ_3 has been replaced by k_3. If in the expressions thus obtained we put ρ_2 equal to k_2, a fixed constant, we get the coördinates of a point whose locus is the asymptotic line in which the surfaces $\rho_3 = k_3$ and $\rho_2 = k_2$ intersect, asymptotic for both surfaces. These expressions may be written

$$(10) \qquad x = l\rho_1 + p, \quad y = m\rho_1 + q, \quad z = n\rho_1 + r,$$

where $l, m, \cdots, r$ are constant along the line; their expressions are readily found from (6). The equations (10) show that the curve along which $\rho_3 = k_3$, and $\rho_2 = k_2$ intersect is a right line—a rectilinear generator for each surface—as we have seen from other considerations.

PRINCETON UNIVERSITY,
February, 1901.

NOTE ON HAMILTON'S DETERMINATION OF IRRATIONAL NUMBERS.

BY DR. H. E. HAWKES.

(Read before the American Mathematical Society, February 23, 1901.)

THE purpose of this note is to call attention to Hamilton's use of the partition (Schnitt) in his definition of certain irrational numbers.[*]

Though Hamilton's method of discussing the irrational number is open to serious criticism, it has some interest as affording an approximation to the rigorous theory afterward developed by Dedekind[†], and also as presenting another example of the natural character of Dedekind's definition of the irrational number.[‡]

For Hamilton time rather than space furnished the rudiment to which his intuition appealed for heuristic purposes and for his conception of unity. Starting with this conception he gives a careful exposition of the operations on positive and negative integers which do not carry one outside the system. A similar discussion of the rational number follows. From the analogy furnished by the laws of operations on integral and rational numbers, Hamilton writes expressions by which he defines the fundamental operations on ratios of any two intervals (steps) of time, and assumes that to any such ratio corresponds a number which he calls an "algebraic number." This he does, even though he has considered the existence of no numbers save integers and fractions. In proceeding to develop his theory of irrational number Hamilton too often makes use of these "algebraic numbers," of whose existence and properties he has proved nothing. Throughout his work, however, fractions could be substituted for "algebraic numbers" and such a change would go far toward making his work rigorous In the following outline of Hamilton's work, it is assumed unless the contrary is stated, that the numbers considered are "algebraic numbers."

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[*] "Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time." *Transactions of the Royal Irish Academy*, vol. 17 (1837), p. 293 This essay was read in 1833

[†] Stetigkeit und irrationale Zahlen, 1872

[‡] In commenting on Tannery's development of the partition (Introduction à la théorie des fonctions d'une variable, 1886) from a remark made by Bertrand (Traité d arithmétique, p 203, 1849) Dedekind says, (Was sind und was sollen die Zahlen, 1893, p xiii): "Diese Uebereinstimmung scheint mir ein erfreulicher Beweis dafur zu sein, dass meine Auffassung der Natur der Sache entspricht"

Hamilton first shows as a corollary of a theorem on multiplication that for any numbers x' and x''

$$x'^2 \lesseqqgtr x''^2 \text{ according as } x' \lesseqqgtr x''.$$

He next assumes that between any two fixed numbers an infinite number of numbers must lie. Suppose A and B be two finite sets of numbers, arranged in order of magnitude, increasing and decreasing respectively, such that the greatest number in A is less than the least number in B. There must exist in general an infinite number of numbers less the least number in B and greater than the greatest number in A. Hamilton goes on to say that if the numbers in A and B are not given explicitly, but by some law, it may happen that no rational number can exist between A and B. But his intuition indicated that such sets could be used to determine non-rational numbers. This he elaborates in detail for a few particular cases, as for example the square root of a positive fraction. His work on this case is in outline as follows:

Let b denote a positive number. Suppose we have the sets A and B composed of an infinite number of rational elements

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots \quad \text{and} \quad \beta_1 > \beta_2 > \beta_3 \cdots,$$

respectively, where

$$\alpha_i^2 < b < \beta_j^2 \qquad (i, j = 1, \cdots).$$

Hamilton shows that these two sets A and B determine uniquely a quantity which he calls the irrational number a He defines the product a^2 by the partition (A', B') where the elements of A' and B' are

$$\alpha_1^2 < \alpha_2^2 < \alpha_3^2 < \cdots \quad \text{and} \quad \beta_1^2 > \beta_2^2 > \beta_3^2 > \cdots,$$

respectively. He then shows that the number afforded by this partition is neither greater nor less than b. Thus he calls

$$a = \sqrt{b}.$$

Hamilton does not define addition or subtraction of partitions, nor does he order them. Neither does he show that every partition of real numbers defines a real number.

YALE UNIVERSITY,
February, 1901.

MUTH'S ELEMENTARTHEILER.

Theorie und Anwendung der Elementartheiler. Von Dr. P. MUTH. Leipzig, B. G. Teubner, 1899. xvi and 236 pages.

THE work under review covers an important subdivision of general invariant theory, a branch which deserves to be more widely known than it is. No doubt the cause of this neglect has been the lack of a text book, and Dr. Muth's monograph will help to remedy the defect.

The original problem which led to the series of investigations in this theory was that of the canonical reduction of two quadratic forms. In 1829 Cauchy published a paper on the secular inequalities of the planets,* in the course of which he showed that two quadratic forms can in general be reduced to sums of squares of the same variables He also proved that the latent roots of the family† are all real, in the special case when one of the quadratic forms is positive for all real (non-zero) values of the variable, *i. e.*, is *definite.* Cauchy's results, though not perfectly general, cover most of the cases which occur in the first stages of geometry and dynamics. Jacobi (1834) found similar results by a somewhat different process.‡ Both Jacobi and Cauchy exclude the possibility of equal latent roots appearing in the problem.

The first systematic account of all possible types of two quadratic forms (allowing for equal latent roots) is to be found in Sylvester's paper (1851) on the contact of lines and surfaces of the second order.§ Here we meet with the idea of classification by means of *invariant factors.*|| Sylvester obtains 4 types of contact for conics, and 12 for quadrics; of course the algebraical possibility that the two forms differ only by a constant factor is trivial in a geometrical

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* Exercices de mathématiques, vol. 4, p. 140 = Oeuvres (2d series), vol. 9, p 174 Cauchy really discusses only the case when one of the original forms is already a sum of squares.

† If A, B are two given quadratic forms, the system of forms $uA + vB$ (u, v arbitrary parameters) will be called a *family* (as an equivalent for Kronecker's term *Schaar*) The *latent roots* are the values of the ratio $(-u : v)$ for which the determinant of the family vanishes; this determinant will be denoted by $|uA + vB|$.

‡ *Crelle*, vol. 12, p. 1 = Ges Werke, vol. 3, p. 191.

§ *Phil. Magazine* (4th series), vol. 1, p. 119

|| Weierstrass's *Elementartheiler*, and Sauvage's *élémentaire diviseur.* A definition is given in a later footnote on Weierstrass's work.

sense.[*] Unfortunately Sylvester did not prove that the invariant factors do constitute a *complete* set of invariants for two quadratic forms; nor did he explain how to effect the reduction of given forms to his standard types.[†]

In 1855 Cayley pointed out that Sylvester's ideas could be applied to the classification of homographies (*i. e.*, one-to-one transformations of space, or collineations).[‡] Here will be found a general formula for the number of possible collineations in space of any dimension n; and the calculation is given for $n = 1, 2, \cdots, 11$. Cayley gives symbols to represent each type by means of the indices of its invariant factors and enumerates the possibilities for $n = 1, 2, 3, 4$.[§] In concluding the paper he says: "*Je reviendrai à cette théorie à une autre occasion*;" but apparently this promise was not fulfilled.

The next advance in the general theory was made in 1858 by Weierstrass, who gave a general method of reducing two quadratic forms to sums of the same squares.[‖] He proved that if one of the forms be definite, the reduction is still possible, even if equalities exist among the latent roots. Using this result in connection with the dynamical problem of small oscillations about a position of equilibrium, Weierstrass showed that the stability is not destroyed by the presence of equal periods in the system; both Lagrange and Laplace supposed that equal periods would involve instability.[¶] Weierstrass's theorem is commonly attributed by English writers[*] to Routh who rediscovered the theorem in his Adams prize essay (1877). Routh's elementary proof of the theorem requires to be supplemented by Weierstrass's algebraical theorem.

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[*] In the algebraical problem we must further allow for the case of no contact, so that in all we have 6 and 14 as the numbers of the distinct types of families of quadratic forms ($n = 3$ or 4).

[†] To illustrate the simplification introduced by the use of invariant factors in geometry, the reader may compare the condition for double contact of two conics as found by Salmon (Conics, p. 346) with Sylvester's; which is simply that there should be two equal *linear* invariant factors of the determinant of the family.

[‡] "Recherches sur les matrices dont les termes sont des fonctions linéaires d'une seule indéterminée." *Crelle*, vol. 50, p. 313 = Coll. Math. Papers, vol. 2, p. 216.

[§] In his list for $n = 4$ the two symbols $\begin{matrix}31\\1\end{matrix}$, $\begin{matrix}21\\21\end{matrix}$ are omitted: Muth would write these [(21)1], [(11)(11)]. This is obviously an oversight on Cayley's part, as he states that 14 types are possible, but gives only 12. The indices of the invariant factors are the differences between Cayley's numbers.

[‖] *Berl. Monatsberichte*, 1858, p. 207 = Ges. Werke, vol. 1, p. 233.

[¶] The point of the theorem will probably be grasped better by considering a special (imaginary) example. Suppose we had a system with kinetic and potential energies, $T = x'y'$, $V = p^2xy + \frac{1}{2}x^2$, then the equations of motion are $x'' = -p^2x$, $y'' = -(p^2y + x)$, and corresponding to a term $A \sin pt$ in x we have one $\frac{At}{2p} \cos pt$ in y. Weierstrass proves that such a case is impossible in a real dynamical system.

A beautiful and elementary proof of Weierstrass's theorem was given by Kronecker[†] in 1868, which appears to be less familiar than it ought to be.

Weierstrass[‡] finally completed the theory of quadratic forms in 1868; and extended his theorems to the more general idea of bilinear forms. He proved that two families of forms are capable of transformation into each other if (and only if) the invariant factors[§] of the two families are the same. The only case of exception to Weierstrass's results occurs when the determinants of the two families are identically zero; and this case was examined by Kronecker in a paper immediately following Weierstrass's (and quoted above). The complete statement of the conditions of equivalence in every case has been effected by Kronecker in a number of memoirs.[||]

Invariant factors presented themselves from another point of view to H. J. S. Smith, who encountered them in connection with linear equations with integral coefficients.[¶] He proves that many of the results known for the invariant factors of a family of forms hold for those of a matrix of integers.

We turn next to the historical development of Cayley's theory of matrices*; the foundations were given by him in 1855 † and his first long memoir on the subject appeared in 1858 ‡; together with an application (in a consecutive paper) to Hermite's theory of the automorphic substitutions of a given quadratic form. Some years later (1867) Laguerre § published an independent account of his calculus of "linear systems" which are virtually Cayley's matrices. But in 1853, Hamilton ‖ had given certain results relating to a "linear and vector" function; which is essentially a matrix of order 3. For instance he proved that such a function satisfies a certain cubic equation with constant coefficients, and he found the reciprocal function. He therefore anticipated Cayley's theorem on matrices ¶ (for the special case $n = 3$). Further, in 1867, he showed (Elements of quaternions) that the cubic may reduce to a quadratic (a special case of Frobenius's theorem referred to below). Probably this is the theory alluded to by Study (in the reference quoted above).

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[*] *E. g.*, Thomson and Tait, Natural Philosophy, vol. 1, § 343 *e* and *m*; where the reader will find some useful remarks on the physical side of the theorem. Routh was led to his theorem by the consideration of linear differential equations with constant coefficients; and in this connection obtained some other results due to Weierstrass, which were given originally in a communication to the Berlin Academy (in 1875), but were only published in vol. 2 of his works.

[†] *Berl. Monatsberichte*, 1868, p. 339 = Ges. Werke, vol. 1, p. 165.

[‡] *Berl. Monatsberichte*, 1868, p 310 = Ges. Werke, vol. 2, p. 19.

[§] *Definition:* Suppose that the determinant of the family $(uA + vB)$ has a repeated factor $(au + bv)^p$; and further that *every* first minor is divisible by $(au + bv)^q$ but not by $(au + bv)^{q+1}$; *every* second minor by $(au + bv)^r$; and so on. Then the numbers $p - q$, $q - r$, $\cdots$ are the indices of the invariant factors to the base $(au + bv)$; they have the property $(p - q) \geqq (q - v) \geqq \cdots \geqq 1$, which was recognized by Cayley (l c., supra). The invariant factors can also be defined rationally, by means of highest common divisors (a remark due to Smith and Kronecker).

[||] For the results (which are rather long) the reader may consult § 8 of Dr. Muth's book; there is another investigation and a list of papers in the *Proc. Lond. Math. Soc.*, vol 32 (1900), p. 98.

[¶] Math. Papers, vol. 1, p 367, and vol. 2, p. 623; some of Smith's results were published before the corresponding theorems for a family. For other references in this direction consult the Encyklopädie der Math. Wiss., vol. 1, pp. 582–597.

In a letter to *Nature* (vol. 44 (1891), p. 79), Professor J. W. Gibbs claims for Grassmann the first suggestion of the theory of matrices (Ausdehnungslehre, 1st ed., 1844). In this connection Whitehead (Universal algebra, Cambridge, 1898, vol. 1, p. 248) refers only to the second edition (1861) of Grassmann's work and implies that Cayley had anticipated Grassmann. I am unable to give any opinion on this point; nor do I know the extent of Grassmann's results.

In 1878 Frobenius** pointed out the important connection between bilinear forms and matrices (or linear substitutions). Being familiar with the results of Weierstrass and Kronecker on the equivalence of families of bilinear forms, †† he was naturally led to introduce the idea of invariant factors into the theory of matrices. This step has proved most fruitful in both theories; it enabled Frobenius to give the first general proof of Cayley's theorem, and to modify the theorem in the case when some of the latent roots of the matrix are equal.* Amongst other methods suggested in this paper is the use of the expansion of $(\lambda E - A)^{-1}$ to investigate properties of A, or rather of the latent roots of A; the expansion is made in powers of $(\lambda - a)$ (a being any latent root), and starts with a term in $(\lambda - a)^{-\alpha}$, $(\lambda - a)^{\alpha}$ being the first invariant factor of $|\lambda E - A|$, which belongs to a.†

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* Study in his report on complex units (Encyklopädie der Math. Wiss., vol. 1, p. 169) states that a theory had been hinted at by Hamilton. (See below.)

† *Crelle*, vol. 50, p. 282 = Coll. Math Papers, vol 2, p. 185.

‡ *Lond. Phil. Trans.*, vol 148, p. 17 = Coll. Math. Papers, vol. 2, p. 475.

§ *Jour. École Polyt.*, vol. 25, p. 215.

‖ Lectures on quaternions, pp. 559–569.

¶ That is: A matrix of order n satisfies an identical equation of order n. Apparently Cayley himself only verified the theorem up to $n = 3$; his proof is for the case $n = 2$ only.

** *Crelle*, vol. 84, p. 1.

†† It may be remarked that Frobenius gives a very convenient summary of these results in §6 of his paper

Among other applications of this method we may mention Frobenius's definition‡ of *any* function of A: provided that $f(A)$ can have a meaning, it is equal to the sum of the residues of $f(\lambda)\ (\lambda E - A)^{-1}$ taken for all the latent roots of A. While speaking of this, I should call attention to what appears to be an oversight in the Clark University decennial volume (Worcester, Mass., 1899); in the report on the mathematical faculty, it is stated that Sylvester (in 1882) was the first to give an expression for the general power of a matrix. But according to Frobenius, Stickelberger (in 1881) gave the expression, and moreover for the case of repeated latent roots; while Sylvester assumes all the latent roots to be unequal.§ For the generalized function of a matrix Sylvester was the first to give a result‖ (but with the same restriction on the latent roots); it is claimed in the Clark volume that the extension to the case of equal latent roots was made by Professor Henry Taber (in 1893,

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* By "latent roots" of a matrix A, we mean the roots of the determinantal equation $|\lambda E - A| = 0$, where E is the unit matrix. Probably the simplest proofs of the theorems are those of Frobenius (*Berl. Sitzungsberichte*, 1896, p 604), though the same proof of Cayley's theorem had been given by Buchheim (*Mess. of Math.*, vol. 13 (1883), p. 62).

† To illustrate the method I collect six theorems on special matrices giving their authors and dates of publication. All the theorems can be proved most simply by Frobenius's method. (*L. R.* denotes latent roots; *I. F*, invariant factors.)

$$A \text{ symmetrical} \begin{cases} L.\ R\ \text{ real (Cauchy, 1829).} \\ I.\ F.\ \text{ linear (Weierstrass, 1850).} \end{cases}$$

$$A \text{ orthogonal} \begin{cases} L.\ R.\ e^{i\phi}\ \text{(Brioschi, 1854).} \\ I.\ F.\ \text{ linear (Frobenius, 1878)} \end{cases}$$

$$A \text{ alternate} \begin{cases} L.\ R.\ \text{ imaginary (Weierstrass, 1879).} \\ I.\ F.\ \text{ linear (Weierstrass, 1879).} \end{cases}$$

All these theorems were rediscovered by Professor Henry Taber (*Proc. Lond. Math. Soc.*, vol. 22 (1891), p. 449).

‡ *Berl. Sitzungsberichte*, 1896, p. 7.

§ After defining $f(A)$ as above, Frobenius (l. c., p. 11) says—"In dieser Weise hat Stickelberger in seiner akademischen Antrittsschrift: *Zur Th. d. lin. Diffgl.* (Leipzig, 1881) die allgemeine Potenz definirt und . . . benutzt. Eine weniger genaue Definition giebt Sylvester: *Sur les puissances et les racines des subst. lin.* (*Comptes rendus*, vol. 94 (1882), p. 55)." Professor Stickelberger informs me (in a private letter) that some copies of his paper were published at the end of 1880.

‖ *Johns Hopkins University Circulars*, vol. 3 (1882); of course the step from the generalized power to the general function is almost self-evident.

1894). But Buchheim[*] (in 1886) had already made the extension, and in a more concise form; both forms are, of course, contained in Frobenius's statement quoted above.

In referring to Frobenius we should remark that he has a claim to be a joint discoverer of a theorem, frequently attributed to Jordan,[†] to the effect that every *periodic* linear substitution can be reduced to the form

$$y_r' = e_r y_r$$

where e_r is a root of unity. This theorem has been recently taken up afresh by E. H. Moore and H. Maschke.[‡] On the whole, the investigation of Frobenius seems the most direct and the clearest of all that I have seen; though, as it requires an elementary acquaintance with the theory of matrices, students of group-theory and differential equations may prefer an independent proof.

Of recent developments, we may refer to a recent paper by S. Kantor[§] (1900) in which the theory of invariant factors of higher kinds (Stufen) is examined by the aid of geometry in space of n^2 dimensions. This appears to be an entirely fresh departure.

Dr. Muth's book is the first published account of the theories which we have sketched above;[||] our thanks are due to the author for having collected so many useful results into one convenient volume. On the other hand it may be questioned if readers beginning the subject will not do well to modify his arrangement. Thus §1, on general properties of invariant factors, will probably be found one of the hardest in the book. In §2 will be found a good account of the theory of "multiplying" bilinear forms; the author follows Frobenius in preference to Cayley,* and introduces a matrix only as a picture (Bild) of the bilinear form. The theory of forms (or systems) with elements which are integers in various regions of rationality, occupies §§3–5 and 18. It may be pointed out that these methods can be applied† to deduce the principal theorems of §1.

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[*] *Phil. Magazine*, 5th series, vol. 22, p. 173.

[†] On p. 16 of his paper, Frobenius shows that for periodic substitutions all the invariant-factors are linear and that the latent roots are roots of unity. On p. 21 it is stated that the reduced forms depend only on the invariant factors (using Weierstrass's results for bilinears), and the combination of these two facts leads directly to the theorem as given. Jordan's statement occurs later in the same volume of *Crelle* (p. 112); there can be little doubt as to their independence.

[‡] *Math. Annalen*, vol. 50 (1898), pp. 215 and 220; cf. L. E. Dickson, "Report on progress in the theory of linear groups," BULLETIN, vol. 6 (1900), p. 13.

[§] *Monatshefte für Math. und Phys.*, vol. 11, p 193; another way of applying n dimensional geometry to matrices has been given by Buchheim (*Proc. Lond. Math. Soc.*, vol. 16 (1885), p. 63) after Grassmann. We may consult also Whitehead's Universal algebra (vol. 1, p. 248).

[||] Two fairly complete accounts of the theory have appeared recently: Ed. Weyr. *Monatshefte fur Math. und Phys.*, vol. 1 (1890), p. 163; Sauvage, *Ann. Ecole Norm. Sup.* (3d series), vol. 8 (1891), p. 285, and vol. 10 (1893), p. 9. Neither of these covers so much ground as Dr. Muth's work For sketches and references we may consult also F. Meyer's reports on general invariant theory; *Jahresbericht der Deutschen Math. Ver.*, vol. 1 (1890), p. 106 and Encyklopädie der Math. Wiss., vol. 1, p. 327.

In §§6–9 the theory of equivalence of families of forms is considered by the methods of Weierstrass and Kronecker; it seems to me that a somewhat easier introduction to the theory will be found in Darboux's paper on quadratic forms;‡ and in Stickelberger's extension of the same to bilinears;§ after which Muth's work will follow. It may be questioned if any advantage is obtained by the use of double suffixes in the reduced forms, at any rate in the tables (pp. 91, 116, 124, 133). The geometrical interpretation of the results on p. 124 by means of conics should be noted; this will be found in Sylvester's paper, already noted.

The remainder of the book is occupied chiefly with special applications; §10 contains Frobenius's method of reducing a given bilinear form by means of congruent substitutions;‖ § 11 gives a method of reducing a linear substitution to Jordan's canonical form, though it seems to me that the most practical method for reducing any substitution (whose latent roots are known) is that due to Jordan himself.¶

An application of the results of §11 is made in §16 to the theory of a system of linear differential equations with constant coefficients; this seems to be due to Weierstrass.

In §§12, 13, we have an account of linear substitutions which are automorphic for a given form; but it ought to be remarked that the first result in this direction is due to

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* It is somewhat remarkable that Dr. Muth nowhere refers to Cayley—not even in his historical account of the subject.

† Cf. Hensel, *Crelle*, vol 114 (1894), pp 25 and 109. Two papers related to this point of view have just been published by Dr. Muth himself, *Crelle*, vol. 122 (1900), pp. 84 and 89.

‡ *Liouville's Jour. de Math*, 2d series, vol. 19 (1874), p. 347.

§ *Crelle*, vol. 86 (1878), p. 20.

‖ This problem was attacked by Kronecker (in 1866) in connection with Weierstrass's general theta functions. Kronecker finally settled the problem in 1874, after some controversy with Jordan. The problem is of interest in certain dynamical questions, as well as in connection with theta functions.

¶ Cours d'Analyse, vol. 3, Art. 143; a reproduction is given in Craig's book on Differential equations A series of papers on this subject will be found in vols. 30-32 of the *Lond. Math. Soc. Proc.* (1899–1900).

Hermite,[*] whose conclusions were translated into the language of matrices by Cayley. No references are given to recent researches on the so-called "singular" cases of automorphic substitutions.[†] It is perhaps worth while to call attention to a result of Loewy's[‡] which shows that, by using a Hermite form (with complex coefficients) in place of the real symmetrical and alternate forms hitherto considered, we avoid all consideration of the singular cases. Loewy's paper contains many other valuable results and will repay careful study.

In §§14, 15, Dr. Muth obtains certain results on linear invariant factors; here we should note Klein's results in his inaugural dissertation (1868)[§], which have been further extended by Loewy.[||] §17 contains an elaborate account of all possible collineations in space of n dimensions, which is afterwards applied to spaces of two and three dimensions. As stated before, the first attempt to classify such transformations is due to Cayley, who did not, however, include the possibility of "singular" collineations; as an illustration of what is meant by such collineations we may take the perspective of ordinary drawing, in which a point of the picture represents all the points on a certain line through the eye. Another form of classification has been given by Whitehead.[¶]

Apparently it is usual to conclude a review with a list of misprints. Of those which I have noticed, not many need delay the reader. In several places the short vertical lines

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[*] *Crelle*, vol. 47 (1854), p. 309 ($n=3$) and *Camb. and Dublin Math. Jour.*, vol. 9 (1854), p. 63 ($n=4$). In connection with Muth's table on p. 172 we may refer to Jordan, *Liouville's Jour. de Math.* (4th series), vol. 4 (1888), p. 349.

[†] See § 11 of Frobenius's paper; it had been considered previously (in the special case $n=3$) by Bachmann, Tannery and Hermite, without the aid of matrices. More recent investigations are those of Loewy in *Math. Annalen*, vol. 48 (1897), p. 97 and vol. 49, p 448; of Taber, *Proc. Lond. Math. Soc.*, vol. 24 (1893), p. 290, and vol. 26 (1895), p. 364; also in *Math. Annalen*, vol. 46 (1895), p 561, and a number of papers in the BULLETIN. In connection with Taber's work we should notice a paper by Rettger (*Amer. Jour. of Math.*, vol. 22 ('900), p. 62, which gives similar results from the point of view of general continuous group theory. (See also the Clark volume, already quoted.)

[‡] *Nova Acta Leopoldina*, vol. 71 (1898), p. 379 = *Math. Annalen*, vol. 50, p. 569. Cf. *Göttinger Nachrichten*. 1900, p 298.

[§] Reprinted in *Math. Annalen*, vol. 23; the theorem alluded to is given on pp. 561, 562 and is, from the point of view of Klein's dissertation, a subsidiary result. In the reprint Klein remarks that the theorem seems not to have been sufficiently considered.

[||] *Crelle*, vol. 122 (1900), p. 53.

[¶] Universal algebra, vol. 1, p. 316.

(used to indicate determinants) have been misplaced. In the lists of reduced forms some small errors have caught my eye;* on p. 142 (footnote) in reference to Frobenius's paper in *Crelle*, vol. 86, we should read p. 146 for p. 20. Near the foot of p. 166, $(S+T)^{-1}\ (S+T)$ should be $(S+T)^{-1}\ (S-T)$; and in some places there are slight errors in the titles of papers quoted

In conclusion I may say that Dr. Muth's book is of great interest and very useful in extending one's knowledge of certain branches of the subject. I hope that it may induce other readers to take up this part of invariant theory, which is important on account of its applications as well as for its intrinsic interest.

T. J. I'A. BROMWICH.

ST. JOHN'S COLLEGE, CAMBRIDGE,
January 1, 1901.

SHORTER NOTICES.

Kurzgefasste Vorlesungen über verschiedene Gebiete der höheren Mathematik, mit Berücksichtigung der Anwendungen. Von DR. ROBERT FRICKE. Large 8vo. Pp. ix + 520. Leipzig, B. G. Teubner, 1900.

Two objects are sought in Dr. Fricke's timely book: first, to supply a defect in German mathematical literature, a handbook for students who have mastered the elements of analysis and are not yet qualified to read profitably the highly specialized treatises; second, to smooth the way for technical students who discover a taste for the more abstract branches of mathematics. The present volume is confined to analysis and theory of functions, a second is announced as in preparation, to treat of advanced portions of algebra and geometry.

The reader is presumed to have a pretty thorough acquaintance with integral calculus, though not with the calculus of imaginaries. Fourier's series are first introduced, with applications to vibrating strings and to diffusion of heat. A short chapter is given to spherical and cylindrical harmonics. with tables for the functions $P_1(\mu)$, $P_2(\mu)$, $-P_6(\mu)$ according to Byerly, and of $J_0(\vartheta)$ and $J_1(\vartheta)$

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* The second form in each of the following needs correction: p. 91 (c. 3); p. 117 (5) and (8). The last of these has an x instead of a y; in each of the other two a suffix has been misprinted.

according to Lommel. The remaining chapters are based upon the third, Functions of a Complex Variable, and it is in this narrow space of 100 pages that the skill of the author in concise and transparent exposition is most strikingly evident.

In attractive, inductive sequence, mainly in geometric style, are presented linear functions and inversion with respect to a circle, stereographic projection, typical rational functions, exponential and hyperbolic functions, then the general concept of analytic functions of the complex variable. The style is that of the lecturer rather than that of the critic, and the requirements of rigor are met by making definitions and hypotheses amply inclusive. At once the motion of an incompressible fluid is taken up briefly, and the question of a function whose integral $\int f(x, y)ds$ from point to point of a domain shall be independent of the path s. Thus are brought in the potential equation, Cauchy's integral, and Green's theorem, while the notion of analytic function is still in the formative stage in the student's mind. Power series, convergence, irregular points, entire functions developed in products, and the gamma function are rapidly sketched, the chapter closing with a well illustrated and suggestive paragraph on Riemann surfaces and the problems arising from them.

Elliptic functions are treated first by the Weierstrassian method, then conversely after Lagrange and Jacobi, as based upon integrals and theta functions. Here, as earlier for spherical harmonics, tables are given: for the elliptic integrals of the first and second kinds and their periods, and for log q as a function of k. These tables are from Lévy's Précis élémentaire etc. Numerous applications of elliptic functions fill 60 pages, among which the short paragraph on confocal quadric surfaces and elliptic coördinates is the one that will be most useful to teachers.

Two concluding chapters are devoted to ordinary and partial differential equations in complex variables, the method and applications being principally drawn from Fuchs, Klein, Schwarz, and Jacobi. These chapters, though condensed, are not crowded, but are throughout readable. Most teachers to whom the purpose indicated by Dr. Fricke appeals as eminently desirable will find this book full of helpful suggestions, and will moreover find their own interest in the successive topics vigorously stimulated by the occasional reading of a chapter. The second volume will be awaited with not a little of curiosity.

Two extracts from the preface are deserving of consideration in this country at least. "— Ich darf wohl auf Zustimmung hoffen, wenn ich die Meinung äussere, dass zu einer tiefer gehenden Verständniss der Mechanik sowohl in ihren Grundproblemen als in ihren Einzelausführungen verschiedene mathematische Disciplinen erforderlich sind, die zur Zeit leider eine regelmässige Pflege an den technischen Hochschulen nicht finden. * * * In Amerika und England sehen wir vielfach ein Unterrichtssystem in Kraft, bei welchem mit relativ geringfügigen mathematischen Hilfsmitteln höchst ausgedehnte Gebiete der Anwendungen berührt werden. Der richtige Weg liegt auch hier gewiss in der Mitte. Um ihn mit Erfolg zu betreten, wird die Mathematik mit den technischen Wissenschaften unter gegenseitiger bereitwilliger Förderung Hand in Hand gehen müssen."

HENRY S. WHITE.

Ebene Geometrie der Lage. Von PROFESSOR DR. R. BÖGER. Leipzig, G. J. Göschen'sche Verlagsbuchhandlung, 1900. (Sammlung Schubert, VII.) 8vo. Pp. x + 289. Price 5 marks.

A NEW text book on the geometry of position is prepared by Dr. Böger, who is Professor at the Real-gymnasium in Hamburg. The book has some novelties, and distinctive merits. In the first place it is well arranged for reference. Section title and number and paragraph number stand at the head of alternate pages. Paragraph numbers appear black in the text, and are set also in smaller type in the outer margin. Definition, theorem, and corollary are distinctly marked as such. Thus the frequent references back and ahead are consulted with the least possible labor, a rapid review is easy, and the student is never left in doubt as to purpose or connection of a sentence. The table of contents is full and well divided, but there is no index. This lack however will not greatly hinder the student, for at every critical point a line of back references is struck which leads to every desired explanation.

As its title indicates, the work is confined to the plane. Beginning with the perspective relation, it concludes with the construction of a polar relation out of any five real or imaginary elementary data (pairs of conjugate points). Steiner and Cremona defined projective ranges to be any two members of a sequence, each member of which is in perspective with the next preceding and the next succeeding. This definition Böger adopts, and so proves as a theorem the invariance of harmonic sets under projective transformation. Von Staudt and Reye chose the reverse order, but I think it probable that most teachers will agree with Böger as to the easier order for students. From two projective pencils or ranges a conic is generated; then follow the simpler properties; afterward the theory of involutions on a range or pencil, then upon a conic; and so are reached the relation of pole and polar and the usual metric properties of diameters and foci.

Here the second part of the work begins, wherein not the conic itself, but the polarity which it mediates, is the principal object. Whatever determines the one determines the other, and these determining elements may just as well be pairs of conjugate points as exclusively self-conjugate points. The problems of construction thus generalized are here solved by the aid of concepts that will be new to many readers: adjoint, conjoint, component, and resultant involutions. On a conic two involutions are *resultant*, for example, when their centers are conjugate with respect to the conic; and if two involutions are considered, both are *component* to their common resultant. The terms seem worthy of general adoption. A brief treatment of sheaves of polarities and of cubic involutions is added, breaking off where the next topic would be polarities of the third order.

Most striking is perhaps the absence of the word imaginary. From the preface we extract the author's reason for the omission. "Von Staudt's theory of imaginary elements, founded upon a normal Wurf, I have replaced by a theory based upon a perfectly arbitrary Wurf. This theory renders it needless to distinguish between real cases and imaginary, because it employs only such proofs as are valid for all cases alike. If we retain the involution that is determined by a Wurf, and base our proofs always on this involution, never on the presence or absence of its two coincidence points, then there will never be need to introduce imaginary elements. Further, the word imaginary is not only needless, it is positively a detriment; for since it has no corresponding image (Vorstellung), its effect can be nothing but confusion. Therefore the word imaginary ought to be outlawed from the geometry of position." This reasoning and a similar plea in the closing paragraph may not convince the reader, but they are well worth study to any one who is prone to mix algebraic short cuts with geometric reasonings.

HENRY S. WHITE.

NOTES.

At the regular meeting of the London mathematical society held on February 14, 1901, the following papers were read: "The distribution of velocity and the equations of the stream lines due to the motion of an ellipsoid in fluid frictionless and viscous," by Mr. T. Stuart; "On factorizable twin binomials," by Lieut. Col. A. J. Cunningham; "Concerning the abelian and related linear groups," by Professor L. E. Dickson; "A geometrical theory of differential equations of the first and second orders," by Mr. R. W. Hudson; "Brocardal properties of some associated triangles," by Mr. R. Tucker; "A note on stability, with a hydrodynamical application," by Mr. T. J. I'A. Bromwich; "Remarks on notation in Lie's theory of continuous groups, and on Schur's determination of a continuous group of given structure, with remarks on Mr. J. E. Campbell's paper (read at the January meeting)," by Mr. H. F. Baker; "Note on curves similar and parallel to one another," by Mr. D. B. Mair.

The publication of the works of Weierstrass, which the Prussian academy of sciences has undertaken, has been unexpectedly delayed, and little progress has been made during the past year. About three-quarters of volume IV. (Abelian functions) has now been printed.

The first number (1901) of the *Annuaire des mathématiciens*, published by Messrs. Carré and Naud, of Paris, is in press, and will soon appear. It will contain about 7,000 names with addresses, a list of the principal mathematical societies, and a list of mathematical periodicals. Besides this, the volume will contain certain scientific notices; among the authors of these notices will probably be Professors P. Appell, Gino Loria, D. Hilbert, F. Klein, C. Méray, J. Petersen, and P. H. Schoute. The preface will be by Dr. C A. Laisant, editor of *L'Enseignement mathématique*, to whom the publication of the volume is due.

The Prussian academy of sciences has received the sum of 100,000 marks from the city of Berlin. The money was given at the celebration of the two hundredth anniversary of the academy, and the interest on it is to be applied at intervals of four years to an important investigation in the field of the natural sciences. The first award is to be made in 1904.

CAMBRIDGE UNIVERSITY.—The Adams prize for 1901 has been awarded to H. M. MACDONALD, fellow of Clare, for an essay on "Electric waves." The subject for the Adams prize in 1903 is: "The bearing on mathematical physics of recent progress in the theory of the representation of discontinuous quantity by series, with special consideration of the logical limitations of the processes involved." The prize is open to the competition of all who have received a degree from the university. The successful candidate will receive about £225. Essays are to be sent to the vice-chancellor by December 16, 1902.

THE Naples academy of mathematical and physical science has awarded its mathematical prize of 1,000 lire for 1899 to Dr. G. TORELLI, professor at Palermo. The subject was the totality of prime numbers. The theme for the next award is the theory of invariants of the ternary biquadratic, considered preferably in relation to the condition for splitting into lower forms. The essays may be written in Italian, French, or Latin, and must be sent in, distinguished by a motto, before March 31, 1902.

THE royal academy of sciences of Turin reannounces the conditions made public January 1st, 1899, for its Bressa prize. The competition is open to the world and closes December 31st, 1902. The prize will be conferred on the discoverer of the most useful invention or the author of the most celebrated work in the mathematical and physical sciences, not excluding geology, history, geography and statistics. The value of the prize is nine thousand six hundred francs.

THE Madrid academy of sciences calls for a historical memoir on the Spanish mathematicians of the sixteenth century for its next annual prize in the mathematical sciences. Biographies and exhaustive accounts of works both published and unpublished are demanded.

UNIVERSITY OF OXFORD. The following announcements are those for the Hilary term 1901 in the mathematical sciences, the courses consisting in general of two lectures per week :— By Professor W. ESSON : Synthetic geometry of conics ; Synthetic geometry of cubics.—By Professor E. B. ELLIOTT : Elements of elliptic functions ; Supplementary lectures on quantics.—By Professor H. H. TURNER : Elementary mathematical astronomy.—By Professor A. E. H. LOVE : Dynamics ; Fourier's series and its physical applications.—By Mr. A. L. DIXON : Calculus of finite differences.—By Mr. J. E. CAMPBELL: Algebra of quantics. —By Mr. P. J. KIRBY: Solid geometry.—By Mr. C. H. THOMPSON: Dynamics of a particle.—By Mr. E. H. HAYES: Geometrical optics.—By Mr. C. E. HASELFOOT: Physical optics.—By Mr. H. T GERRANS: Hydrodynamics. By Mr. J. W. RUSSELL: Pure geometry.—By Mr. C. LEUDESDORF: Geometry (Maxima and minima, inversion, etc.).

UNIVERSITY OF PARIS. The following courses in the mathematical sciences are announced for the second semester at the faculty of sciences, the semester opening March 1st, 1901, and each course consisting of two lectures per week :—By Professor É. PICARD: Algebraic functions and the transcendents associated with them. The introductory lecture to the course is devoted to the scientific work of Hermite; this lecture will appear in the current volume of the *Annales de l'École Normale.*—By Professor E. GOURSAT: Differential equations.—By Professor P. APPELL: Continuation of the course of the first semester in mechanics.—By Professor J. BOUSSINESQ: Continuation of the course of the preceding semester in mathematical physics.—By Professor G. KOENIGS: Study of machines. —By M. C. BOURLET, replacing Professor L. RAFFY: Differential equations and their applications to mechanics and physics.—By Professor J. HADAMARD and M. M. SERVANT: Conferences on infinitesimal calculus.—By Professor P. PUISEUX: Conferences on mechanics and astronomy.—By Professor J. HADAMARD and MM. H. ANDOYER and E. BLUTEL: Conferences on the subjects for the agrégation.

THE several German universities below offer during the summer semester 1901 courses in mathematics as follows:

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Plane analytic geometry, four hours; Theory of determinants, two hours; Seminar, one hour.—By Professor R. STURM: Theory of geometrical relationship, part one, three hours; Higher algebraic curves, three hours; Seminar on descriptive geometry, two hours.—By Professor J. FRANZ: The problem of three bodies, two hours; Calculus of finite differences, with applications to special perturbations of the heavenly bodies, four hours.—By DR. F. LONDON: Definite integrals, three hours; Mathematics of insurance, two hours.

UNIVERSITY OF ERLANGEN.—By Professor P. GORDAN: Theory of differential equations, four hours; Theory of invariants, four hours; Seminar, three hours.—By Professor M. NOETHER; Differential and integral calculus, II, four hours; Descriptive geometry, with exercises, two hours; Selected chapters in mechanics, two hours.

UNIVERSITY OF FREIBURG.—By Professor J. LÜROTH: Theory of functions, five hours; Theoretical astronomy, two hours.—By Professor L. STICKELBERGER: Integral calculus, four hours; Higher plane curves, three hours; Seminar.—By Dr. A. LOEWY: Algebraic equations, four hours; Foundations of geometry, two hours.—By Dr. E. REBMANN: Elements of projective geometry, two hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Foundations of geometry, four hours; Elements of algebra, two hours; Mathematical seminar, one hour.—By Professor E. NETTO: Plane analytic geometry, four hours; Introduction to the theory of functions, two hours; Seminar, one hour.—By Professor R. HAUSSNER: Differential geometry, with exercises, five hours; Technical mechanics, second part, with exercises, four and one half hours.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Mechanics II, four hours; Theory of potential, four hours; Seminar, two hours—By Professor E. STUDY: Analytic geometry I, four hours; Introduction to synthetic geomtry, two hours; Select chapters in the theory of functions, one hour; Seminar, one hour.

UNIVERSITY OF HEIDELBERG.—By Professor L. KOENIGSBERGER: Infinitesimal calculus, four hours; Theory of functions, four hours; Seminar, two hours.—By Professor M. CANTOR: Algebraic analysis, four hours; Arithmetic and algebra, three hours.—By Professor K. KOEHLER, Analytic geometry, three hours.—By Professor G. LANDSBERG: Theory of determinants, two hours; Algebraic curves, two hours.—By Dr. K. BOEHM: Introduction to the theory of continuous transformation groups, with applications to geometry and differential equations, one or two hours; Gauss's memoir on general theory of forces attracting or repelling as the inverse square of distance, one hour.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Differential equations, four hours; Plane analytic geometry, four hours.—By Professor A. GUTZMER: Differential calculus, five hours; Theory of potential, four hours; Seminar.—By Professor G. FREGE: The foundations of arithmetic, four hours; Mathematical evercises, two hours.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Introduction to algebra and determinants, four hours; Theory of elliptic functions, four hours; Seminar, one hour. —By Professor P. STÄCKEL: Infinitesimal calculus and introduction to analysis, four hours; Advanced dynamics of rigid bodies, four hours; History of mathematics since the Renaissance, one hour; Seminar, one hour.

UNIVERSITY OF KOENIGSBERG.—By Professor F. MEYER: Introduction to the theory of numbers, four hours; Seminar, one hour.—By Professor A. SCHOENFLIES: Differential geometry of curves and surfaces, four hours; Seminar, one hour.—By Professor L. SAALSCHÜTZ: Differential calculus, four hours, with seminar one hour; Gauss's trigonometric and other interesting series, two hours.—By Dr. J. RAHTS: Higher geodesy, two hours; Least squares, two hours.—By Dr. T. VAHLEN: Analytic geometry of the plane and of space, four hours, with exercises, one hour.—By Dr. E. MÜLLER: Introduction to descriptive geometry, two hours.

UNIVERSITY OF LEIPSIC: By Professor C. NEUMANN: Theory of potential and spherical harmonics, four hours; Seminar, two hours.—Professor A. MAYER: Courses to be announced later—By Professor O. HÖLDER: Ordinary differential equations, four hours; Scientifically rigorous foundation of arithmetic, one hour; Seminar, one hour.—By Professor F. ENGEL: Analytic geometry of the plane and of space, four hours; Differential invariants (continued), one hour; Projective geometry and theory of invariants (continuation of projective geometry), two hours; Seminar, with Dr. G. Kowalewski, one hour.—By Dr. F. HAUSDORFF: Differential geometry, four hours; Mengenlehre, two hours.—By Dr. G. KOWALEWSKI; Algebraic analysis, four hours, with exercises, one hour; Line geometry, two hours; Seminar on differential invariants with Professor Engel, one hour.—By Dr. H. LIEBMANN: Algebraic equations, two hours; Synthetic geometry of conics, two hours; Selected chapters in the theory of numbers, one hour.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY: Infinitesimal calculus, four hours; Calculus of variations, two hours; Seminar, four hours.—By Professor E. HESS: Analytic and synthetic treatment of plane geometry, four hours; Integration of differential equations, three hours; Selected chapters on applied mechanics, two hours; Seminar, four hours.—By Dr. F. v. DALWIGK: Introduction to mathematics for chemists, three hours, with exercises, one hour; Collineation and perspective, two hours.

University of Strassburg.—By Professor T. Reye: Selected chapters on higher synthetic geometry, three hours; Theory of forces which attract according to Newton's law, (theory of potential), three hours; Seminar, two hours.—By Professor H. Weber: Theory of abelian functions, four hours; Applications of elliptic functions to mechanics and physics, two hours; Seminar, two hours.—By Professor G. Roth: Differential and integral calculus, three hours, with exercises, two hours; Plane analytic geometry, three hours—By Professor A. Krazer: Definite integrals, three hours; Analytic geometry of space, three hours; Theory of the theta functions, two hours; Seminar, one hour.—By Dr. E. Timerding: Theory of surfaces, two hours.—By Dr. J. Wellstein: Descriptive geometry, II, two hours.

University of Tübingen.—By Professor A. v. Brill: Analytic geometry of space, three hours; Theory of curvature of surfaces, four hours; Seminar, in two sections, two hours.—By Professor H. Stahl: Elementary analysis, three hours, with exercises, one hour; Higher analysis, I (differential calculus), four hours, with exercises, one hour.—By Professor L. Maurer: Synthetic geometry, two hours, with exercises, one hour; Descriptive geometry, I, two hours, with exercises, two hours.

University of Würzburg.—By Professor F. Prym: Integral calculus, six hours, with exercises, two hours; Seminar on topics in the theory of functions, two hours.—By Professor A. Voss: Analytic and synthetic geometry of conics, four hours; Analytic mechanics, four hours; Seminar on topics in higher mathematics, two hours.—By Professor E. Selling: Partial differential equations of mathematical physics, four hours; Theory of planetary motions, three hours.

University of Zurich.—By Professor H. Burkhardt: Algebraic analysis, four hours; Partial differential equations, four hours; Seminar, two hours.—By Professor A. Weiler: Analytic geometry, II, two hours; Descriptive geometry, II, two hours; Map projection, two hours; Synthetic geometry, II, two hours.

The dedication of the monument to Francesco Brioschi took place on December 13, 1900, the third anniversary of his death. The monument, which is of bronze, has been placed in the Royal Technical Institute of Milan. It is the work of the sculptor L. Secchi, of Milan, and was erected with a part of the money collected by subscription from the friends, admirers, and pupils of BRIOSCHI. At the inauguration addresses were made by Professors G. COLOMBO, P. BLASERNA, G. CELORIA, and G. BARDELLI. These addresses are published in the *Annali di Matematica* for January, 1901 (3d series, volume V., page 141).

OWING to the state of his health, Professor P. G. TAIT of the chair of natural philosophy in the University of Edinburgh, has intimated his approaching resignation, after a distinguished service of over forty years.

DR. J. W. L. GLAISHER has been elected president of the Royal astronomical society of Great Britain for the forthcoming year. The honorary secretaries are Messrs. F. W. DYSON and E. T. WHITTAKER, and the foreign secretary is, as before, Sir WILLIAM HUGGINS The gold medal of the society was this year awarded to Professor E. C. PICKERING, of Harvard University.

PROFESSOR A. PRINGSHEIM has been promoted to a full professorship in the University of Munich.

PROFESSOR M. CANTOR has been chosen corresponding member of the Royal institute of science, letters, and art of Venice.

IN justice to Professor C. N. LITTLE, whose resignation from the faculty of the Leland Stanford Jr. University was noted in the preceding number of the BULLETIN, it is proper to state that Professor Little's resignation was tendered voluntarily as an expression of disapproval of the administrative policy of the university.

PROFESSOR D. E. SMITH, principal of the Brockport, N. Y., State normal school, has accepted a call to the professorship of mathematics in Teachers College, Columbia University, and will enter upon the duties of his new position at the beginning of the next academic year.

MISS E. F. PENDLETON, instructor in mathematics at Wellesley College, has been elected dean of the college. Miss HELEN MERRILL has been promoted to an associate professorship of mathematics. Miss R. H. VIVIAN, alumnæ fellow in mathematics at the University of Pennsylvania, has been appointed instructor in mathematics at Wellesley.

Dr. G. A. Miller, of Cornell University, has been appointed to an assistant professorship of mathematics at Leland Stanford University.

Dr. O. Schlömilch, formerly professor of mathematics at the Technical School at Dresden, died on February 7th, at the age of seventy-eight years.

Professor Charles McDonald, died at Halifax, N. S., on March 10th, after a service of over thirty years in Dalhousie College.

Dr. J. M. Rice died at Northboro, Mass., on March 2d, aged sixty-eight years. He was appointed professor of mathematics at the Naval academy at Annapolis in 1870.

Professor G. P. Starkweather, of Yale University, died on March 21st, at the age of twenty-eight years.

The library of the late Professor E. B. Christoffel is offered for sale by Gustav Fock of Leipzig.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Beltrami (L.). Francesco Brioschi: nel giorno della morte; un anno dopo; davanti al monumento, dicembre 1900. Milano, Allegretti, 1900. 8vo. 36 pp., portrait.

Bôcher (M.). The theory of linear dependence. Salem, Mass., Salem Press, 1901. 4to. 16 pp. (From the *Annals of Mathematics*, second series Vol. 2.)

Borini (B.). I continuanti. Forli, Medri, 1900. 4to. 117 pp.

Bosworth (A. L.). Begründung einer vom Parallelenaxiome unabhängigen Streckenrechnung. (Diss.) Gottingen, Dietrich, 1900. 8vo. 57 pp.

Gmeiner (J. A.). See Stolz (O.).

Halsted (G. B.). Gauss and the non-euclidean geometry. 11 pp. (From *Science*, new series, Vol. 12, pp. 842–846.)

Haentzschel (E.). Über die verschiedenen Grundlegungen in der Trigonometrie. (Progr.) Berlin, 1900. 4to. 31 pp.

HENTSCHEL (O.). Ausfuhrung einiger konformen Abbildungen der Parabel auf den Kreis und unendlich lange Parallelstreifen. Leipzig, Fock, 1900 4to. 8 + 4 pp., 9 plates. M. 1.20

HERZ (N.). Wahrscheinlichkeits- und Ausgleichungsrechnung. Leipzig, Goschen, 1900. 8vo. 4 + 381 pp. Cloth. (Sammlung Schubert, No. XIX) M. 8.00

HILBERT (K. S.). Das allgemeine quadratische Reciprocitatsgesetz in ausgewahlten Kreiskorpern der zweiten Einheitswurzeln. (Diss.) Göttingen, 1900. 8vo. 72 pp.

HUGI (H R.). Begleitkurven eines Punktes in Bezug auf eine Kurve zweiter Ordnung. (Diss.) Bern, 1900. 4to.. 35 pp., 1 plate.

KORN (A.). Abhandlungen zur Potentialtheorie. Teil I: Allgemeiner Beweis der Methoden des alternierenden Verfahrens und der Existenz der Lösungen des Dirichlet'schen Problems im Raume. Teil II: Eine weitere Verallgemeinerung der Methode des arithmetischen Mittels. Berlin, 1901. 8vo. 64 pp. M. 2 00

KURTZ (E.). Das Netz der Kegelschnitte, die ein gegebenes Poldreieck haben. (Diss.) Münster, 1900. 8vo. 23 pp.

LAMPART (E.). Die geodätischen Linien auf der dreiaxigen Flache zweiten Grades, welche sich mittels einer Transformation zweiten Grades durch elliptische Funktionen ausdrücken. (Diss.) München, 1900. 8vo. 42 pp. M. 3 00

LEM (J. W.). Analytische Theorie der Ruimtekrommen Leiden, 1899. 4to. 3 + 136 pp.

MÄNNEL (W.). Der geometrische Calcul. (Progr) Prag, 1900. 8vo. 37 pp.

MATTER (K.). Die den Bernoulli'schen Zahlen analogen Zahlen im Korper der dritten Einheitswurzeln. (Diss.) Zurich, 1900. 8vo. 38 pp.

MOLNÁR (E.). Bestimmung der zweiten Ableitungen der Flächenpotentiale. (Diss.) Zurich, 1900. 8vo. 68 pp.

MOREAU (L.) Analyse ou nombre de solutions et fixations des racines remarquables de l'equation $a^x = x$. Bruxelles, 1900. 8vo. 16 pp M. 1.00

MÜLLER (E.). Ueber die Algebra der Logik. Die Grundlagen des Gebietekalkuls. (Progr.) Tauberbischofsheim, 1900. 4to. 30 pp. M. 1.50

RICCI (G.). Lezioni di algebra complementare. Verona, 1900. 8vo. 466 pp. M. 8.50

ROSSI (D. L). Teoria generale della parabola e della catenaria; appunti di analisi matematica che possono essere utili nello studio della catenaria telegrafica. Cagliari; Tipografia Commerciale, 1900. 8vo 77 pp., 3 plates. Fr. 2.00

RUDERT (E.). Über kleine Kugelkreise. Eine Anwendung von Grassmanns Ausdehnungslehre. (Diss.) Leipzig, 1900. 4to. 33 pp.

SAILER (E.). Die Aufgaben aus der Differential- und Integralrechnung und aus der analytischen Geometrie, welche bei der Prüfung für das Lehramt der Mathematik und Physik an den k. bayerischen humanistischen und technischen Unterrichtsanstalten 1873–93 gestellt wurden. München, 1900. 8vo. 187 pp. M. 4.80

SCHLESINGER (L.). Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variabeln. Leipzig, Göschen, 1900. 8vo. 8 + 309 pp. Cloth. (Sammlung Schubert, No. XIII.) M. 8.00

SIMON (M.). Analytische Geometrie des Raumes. Teil I: Gerade, Ebene, Kugel. Leipzig, Göschen, 1900. 8vo 3 + 152 pp. Cloth. (Sammlung Schubert, No. IX) M. 4.00

——. Teil II: Die Flächen zweiten Grades. Leipzig, Göschen, 1901. 8vo. 4 + 176 pp. Cloth. (Sammlung Schubert, No. XXV.) M. 4.40

STOLZ (O.) und GMEINER (J. A.). Theoretische Arithmetik. Abteilung I: Allegemeines; die Lehre von den rationalen Zahlen. 2te Auflage der Abschnitte I–IV des ersten Teiles der Vorlesungen über allgemeine Arithmetik von O. Stolz. Leipzig, Teubner, 1900. 8vo 4 + 98 pp. (Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. IV.) M. 2.40

WENDLER (A.). Ueber die Flächen, welche dem partikularen Integrale der Differentialgleichung $\frac{\partial^2 z}{\partial x \partial y} = 0$ entsprechen. (Diss.) München, Reinhardt, 1900. 8vo. 48 pp., 1 plate. M. 1.50

WORM (H.). In der Ebene einem gegebenen Viereck ein Viereck von kleinstem Umfange einzubeschreiben. (Diss.) Leipzig, 1900. 4to. 47 pp.

II. ELEMENTARY MATHEMATICS.

ALASIA (C.). Esercizi ed applicazioni di trigonometria piana, con 400 esercizi e problemi. Milano, 1900. 12mo. 308 pp. Fr. 2.00

BALTZER (R.). Elementi di matematica. Tradotti da L. Cremona. Parte VI: Trigonometria. 5ta edizione. Genova, 1900. 8vo. Fr. 2.50

BREMIKER (C.). Logarithmisch-trigonometrische Tafeln mit 6 Decimalstellen. Neu bearbeitet von Th. Albrecht. 13te Ausgabe. Berlin, 1901. 8vo. 18 + 598 pp. M. 4.20

FOURREY (E.). Récréations arithmétiques. 2e édition. Paris, Nony, 1901. 8vo. 8 + 263 pp.

GAMBIOLI (D.). See RIBONI (G.).

GRÉVY (A.). Géométrie à l'usage des classes de lettres. Géométrie dans l'espace (classes de seconde et de rhétorique). 2e édition. Paris, Nony, 1901. 18mo. 140 pp.

——. Géométrie à l'usage des classes de lettres. Géométrie plane (classes de quatrième et de troisième). 2e édition. Paris, Nony, 1901. 18mo. 200 pp.

LIGOWSKI (W.) Sammlung funfstelliger logarithmischer, trigonometrischer und nautischer Tafeln nebst Erklärungen und Formeln der Astronomie. 4te Auflage. Kiel, 1900. 8vo. 28 pp. Boards. M. 1.00

MARTUS (H. C. F.) Mathematische Aufgaben zum Gebrauche in den obersten Klassen höherer Lehranstalten. Aus den bei Reifeprufungen an deutschen hoheren Schulen gestellten Aufgaben ausgewählt und mit Hinzufugung der Ergebnisse zu einem Uebungsbuche vereint. Teil 4: Ergebnisse der Aufgaben des 3ten Teils. Dresden, Koch, 1901. 8vo. 182 pp. Cloth. M. 4.50

PFLIEGER (W.). Elementare Planimetrie. Leipzig, Göschen, 1901. 8vo. 7 + 430 pp. Cloth. (Sammlung Schubert, No. II.) M 4.80

RHEUDE (F.). Lösungen zu den mathematischen Absolutorialaufgaben der bayerischen Realschulen (1869–1900 inclusive). 2te Auflage. München, 1900. 8vo. 192 pp. M. 2.20

RIBONI (G.). Elementi di geometria a uso delle scuole secondarie inferiori, corredati da una raccolta di circa novecento esercizi per cura di D. Gambioli. 4ta edizone con aggiunte e migliorata. Bologna, Zanichelli, 1901. 16mo. 8 + 396 pp. Fr. 2.00

RUNGE (C.). Praxis der Gleichungen. Leipzig, Göschen, 1900. 8vo. 3 + 196 pp. Cloth. (Sammlung Schubert, No. XIV.) M. 5.20

SCHMID (K.). 100 ausführlich geloste arithmetische Aufgaben der bayerischen Lehrer-Anstellungsprüfungen 2te Auflage. München, 1900. 8vo. 112 pp. M. 1.20

TESTI (G. M.). Elementi di geometria ad uso delle scuole tecniche. 5ta edizione in gran parte modificata e messa in armonia con gli ultimi programmi governativi. Livorno, Giusti, 1901. 16mo. 7+205 pp. Fr. 1.75

WORTNER (F.). Geometrie und geometrisches Zeichnen fur Knabenbürgerschulen. Wien, Tempsky, 1900. 8vo. 7 + 170 pp. Cloth. M. 1.80

III. APPLIED MATHEMATICS.

BENTON (J. R.). Abhangigkeit des specifischen Torsionswiderstandes einiger Metalldrähte von der Spannung. (Diss.) Leipzig, Barth, 1900. 8vo. 37 pp.

BLONDIN (J.). See POINCARÉ (H.).

BOVEY (H. T.). Strength of material and theory of structures. 3d revised edition. New York, Wiley, 1900. 8vo. 17 + 835 pp. Cloth. $7.50

CLAUZEL. Théorie du navire. Partie I: Equilibre et stabilité du navire en eau calme. Paris, 1900. 8vo. With atlas of 23 tables and 56 plates 4to. Fr. 20.00

DU BOIS (A. J.). Stresses in framed structures including "The strength of materials" and "Theory of flexure," etc. New York, Wiley, 1900. 26 + 617 pp. Cloth. 4to. $10.00

ERNST (A.). Eingriffverhältnisse der Schneckengetriebe mit Evolventen- und Cykloidenverzahnung und ihr Einfluss auf die Lebensdauer der Triebwerke. Ein Abriss der graphischen Untersuchung von

Schneckenräderwerken für die Praxis und den Unterricht an technischen Lehranstalten. Berlin, Springer, 1901. 8vo. 6 + 82 pp., 17 plates. Cloth. M. 4.00

FLAMAND (A.). Hydraulique. 2e édition, considérablement augmentée. Paris, Béranger, 1900. 8vo. 15 + 689 pp. (Encyclopédie des travaux publics.) Fr. 26.00

FLEMING (J. A.). The alternate current transformer in theory and practice. Vol. I: Induction of electric currents. 3d edition. London, The Electrician, 1901. 8vo. 628 pp Cloth. 12s. 6d.

FÖPPL (A.). Résistance des matériaux et éléments de la théorie mathématique de l'élasticité Traduit de l'allemand par E Hahn. Paris, Gauthier-Villars. 1901. 8vo. 493 pp. (Encyclopédie industrielle.)

GROSS (T.). Kritische Beitrage zur Energetik. Teil I: Die Verwandlungen der Kraft nach Robert Mayer. Berlin, Krayn, 1901. 8vo. 18 + 58 pp. M. 1.75

GROSSMANN (L). Die Mathematik im Dienste der Nationalökonomie unter Rücksichtnahme auf die praktische Handhabung der Disciplinen der Finanzwissenschaft und Versicherungstechnik mit einigen, durch selbständige wissenschaftliche Errungenschaften auf dem Gebiete der reinen Mathematik begründeten, neuen Fundamenten der politischen Arithmetik, für Versicherungs- und Bank-Institute, sowie auch Lehrkräfte höherer Bildungsanstalten besonders geeignet. 12te (Schluss-)Lieferung. Wien, Grossmann, 1900. 8vo. 8 + 80 pp., 1 plate. M. 5.00

GROTH (H.). Zur Dynamik des Himmels. Hamburg, Laeisz, 1901. 8vo. 4 + 74 pp. M. 3.00

HAHN (E.). See FÖPPL (A.).

J. (F.). Eléments de mécanique, avec de nombreux exercices. 4e édition, répondant aux programmes des classes de mathématiques élémentaires et de première-sciences. Paris, Poussielgue, 1897. 16mo. 8 + 319 pp. (Cours de mathématiques élémentaires.)

JOHNSON (J. B.). Materials of construction; treatise for engineers on the strength of engineering materials. 3d revised enlarged edition. New York, Wiley, 1900. 15 + 795 pp. 8vo. $6.00

JULIUSBURGER (P.). Ueber das Dupré-Rankine'sche Dampfspannungsgesetz. (Diss.) München, 1900. 8vo. 131 pp., 3 plates. M. 5.00

LORENZ (H.). Dynamik der Kurbelgetriebe mit besonderer Berücksichtigung der Schiffsmaschinen. Leipzig, Teubner, 1901. 8vo. 5 + 156 pp. M. 5.00

MAYER (A). Die Gleichgewichtsbedingungen reibungsloser Punktsysteme und die verschiedenen Arten des Gleichgewichts. (Progr.) Leipzig, 1899. 4to. 23 pp.

MAINKA (C.). Untersuchung über die Verlängerung des Mondkorpers nach der Erde zu. (Diss.) Breslau, 1900. Folio. 29 pp., 1 plate.

MERRIMAN (M.). Text-book on mechanics of materials 9th edition. New York, Wiley, 1900. 10 + 368 pp. 8vo. Cloth. $4.00

MOLL (D. P). Het Princip van Huygens. Utrecht, 1900. 8vo. 4 + 133 pp. Boards.

NÉOULCÉA (E.). See POINCARÉ (H.).

Neugebauer (P. V.). Ein Beitrag zur Theorie der speciellen Störungen mit Anwendung auf eine Verbesserung der Bahn des Planeten Philomela. (Diss.) Breslau, 1901. Folio. 48 pp.

Ostenfeld (A.). Teknisk Statik. Grundlag for Forelaesninger paa Polyteknisk Laereanstalt. Del I. Kjöbenhavn, 1900. 8vo. 500 pp., 33 plates. M. 12 60

Planck (M.). Lectures on thermodynamics. Russian translation by Katerinof. St. Petersburg, 1900. 8vo. 239 pp M. 7.20

Plank (F.). Lehrbuch der politischen Arithmetik. Leipzig, 1901. 8vo. 174 pp. Cloth. M. 2.75

Poincaré (H.). Electricité et optique; la lumière et les théories électrodynamiques Leçons professées à la Sorbonne en 1888, 1890 et 1899. 2e édition, revue et complétée par J Blondin et E. Néculcéa. Paris, Carré et Naud, 1901. 8vo. 10 + 647 pp. (Cours de physique mathématique.)

Schiller (N.). Einige thermodynamisch abzuleitenden Beziehungen zwischen den Grossen, die den physikalischen Zustand einer Lösung characterisiren. 8vo. 30 pp. (From *Archives néerlandaises des sciences exactes et naturelles.*)

Strutt (J. W.) [Baron Rayleigh]. Scientific papers. Vol. II: 1881–1887. London and New York, Macmillan, 1901. 8vo. 14 + 598 pp. Cloth. (Cambridge University Press series.) $5.00

Thierry (D. R.). Over de toepassing van het Theorema van Fourier in de Theorie der Buigingsverschijnselen. Utrecht, 1900. 8vo. 3 + 89 pp.

Weinstein (B.). Thermodynamik und Kinetik der Körper. Vol. I: Allgemeine Thermodynamik und Kinetik und Theorie der idealen und wirklichen Gase und Dämpfe. Braunschweig, Vieweg, 1901. 8vo. 18 + 484 pp. M 12 00

NON-OSCILLATORY LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society February 23, 1901.)

WE shall be concerned with the differential equation

$$(1) \qquad \frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

and for the sake of simplicity we will assume that the coefficients p and q are, throughout the finite interval $a \leqq x \leqq b$, continuous real functions of the real variable x. We shall find it convenient to lay down the following definition :

The equation (1) is said to be oscillatory or non-oscillatory in the interval $a \leqq x \leqq b$ according as it does or does not have at least one solution (not identically zero) which vanishes more than once in this interval.

It is my object in the present paper to deduce certain conditions (chiefly sufficient conditions) that the equation (1) should be non-oscillatory. Such conditions have been obtained by Picard (Traité d'analyse, volume III, pp. 101–104); but the method which I use is not only entirely different and, as it seems to me, less artificial than that of Picard, but yields, besides all of Picard's results, others which Picard's method does not give.

My starting point is the special case $p = 0$:

$$(2) \qquad \frac{d^2y}{dx^2} + qy = 0.$$

Equation (2) is non-oscillatory in the interval $a \leqq x \leqq b$ if throughout this interval $q \leqq 0$.

For if (2) has a solution y which vanishes more than once in the interval in question, let x_1 and x_2 be two successive roots of y. We may, without loss of generality, assume $y > 0$ when $x_1 < x < x_2$, as, if this were not the case, we could replace y by $y_1 = -y$. We have then $y'(x_1) > 0$, $y'(x_2) < 0$; but by the law of the mean

$$y'(x_2) - y'(x_1) = (x_2 - x_1)\,y''(\xi) \qquad (x_1 < \xi < x_2).$$

Accordingly $y''(\xi) < 0$. This, however, is impossible, since by equation (2) $y''(\xi) = -q(\xi)\cdot y(\xi)$.

By reducing (1) to the binomial form (2), we shall get a theorem concerning (1) similar to the one just proved for (2). This reduction is most commonly performed by means of a change of dependent variable y.* It can, however, equally well be performed by changing the independent variable x, or, more generally, by changing both independent and dependent variable. We will consider at once this general transformation

$$(3) \qquad t = f(x), \quad y = \varphi(x)\,\bar{y}.$$

We assume here that f and φ have continuous first and second derivatives throughout the interval $a \leqq x \leqq b$. Furthermore, since we do not wish the solutions of the transformed equation to become infinite, we assume that φ does not vanish, say for distinctness

$$\varphi(x) > 0 \qquad (a \leqq x \leqq b).$$

Finally, since we wish the interval $a \leqq x \leqq b$ to correspond in a one to one manner to an interval on the t-axis, we assume that f' does not vanish in the interval $a \leqq x \leqq b$.

A peculiarity of this transformation which makes it available for our purposes may be stated as follows:

The oscillatory or non-oscillatory character of equation (1) *is invariant with regard to transformations* (3).

The transformation (3) carries (1) over into

$$(4) \quad \frac{d^2\bar{y}}{dt^2} + \frac{1}{f'}\left[\frac{f''}{f'} + 2\frac{\varphi'}{\varphi} + p\right]\frac{d\bar{y}}{dt} + \frac{\varphi'' + p\varphi' + q\varphi}{\varphi\cdot(f')^2}\,\bar{y} = 0,$$

accents denoting differentiation with regard to x.

Choosing φ at pleasure, subject to the restrictions above mentioned, let us determine f so that the second term of (4) drops out,

$$(5) \qquad f = k\int_c^x \varphi^{-2} e^{-\int_c^x p\,dx}\,dx + k' \qquad (k \neq 0),$$

where c is any point of the interval $a \leqq x \leqq b$.

It is important to notice that all the conditions which f was to fulfill are satisfied by this function.

Equation (4) now reduces to

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*See, *e. g*, Forsyth's Treatise on differential equations, p. 88. I note in passing that this reduction is possible only if the coefficient p has a continuous first derivative, a restriction which need not be imposed if we reduce to the binomial form by a change of independent variable.

$$(6)\qquad \frac{d^2\bar{y}}{dt^2}+\frac{\varphi^3}{k^2}e^{2\int_c^x p\,dx}(\varphi''+p\varphi'+q\varphi)\bar{y}=0.$$

When we apply to this equation the theorem proved concerning equation (2), we see that equation (6), and accordingly also equation (1), will be non-oscillatory provided $\varphi''+p\varphi'+q\varphi\leqq 0$. That is

If a function φ exists which, together with its first and second derivatives, is continuous throughout the interval $a\leqq x\leqq b$, and which satisfies the two conditions

$$(7)\qquad \varphi>0 \qquad (a\leqq x\leqq b),$$

$$(8)\qquad \varphi''+p\varphi'+q\varphi\leqq 0 \qquad (a\leqq x\leqq b),$$

then (1) *is non-oscillatory in this interval.**

By assuming for φ special functions we can obtain useful and easily applied criteria for proving that special equations of the form (1) are non-oscillatory. We add a few such criteria, noting to the left the function φ used, and to the right the special conditions, if any, which must be satisfied if the formula is to be applied. In these formulæ m and a denote constants to which we may assign any real values we please.

(a) $\varphi=1,$ $q\leqq 0.$

(b) $\varphi=e^{mx},$ $q\leqq -mp-m^2.$

(c) $\varphi=x^{-m},$ $q\leqq\dfrac{mxp-m(m+1)}{x^2}$ $(0<a).$

(d) $\varphi=m-x,$ $q\leqq\dfrac{p}{m-x}$ $(b<m).$

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* By letting $\phi=e^{-\int_0^x\lambda dx}$ we obtain the following theorem :

If a function λ exists which, together with its first derivative, is continuous throughout the interval $a\leqq x\leqq b$, and which satisfies the condition

$$(8')\qquad q\leqq\lambda'-\lambda^2+\lambda p \qquad (a\leqq x\leqq b),$$

then (1) *is non-oscillating in this interval.*

Conversely, the theorem of the text follows from this one ; so that the two theorems are precisely equivalent to each other.

Picard (1 c., p. 102) deduces by another method a theorem identical with the one just stated, except that the inequality (8′) is replaced by

$$q\leqq\lambda'-(\lambda-p/2)^2.$$

This inequality is always satisfied when (8′) is satisfied, but the converse is not true. It is only when $p=0$ that Picard's result is as general as ours.

$$(e) \quad \varphi = m^2 - x^2, \qquad q \leqq \frac{2(1+xp)}{m^2 - x^2} \quad (-|m| < a < b < |m|).$$

$$(f) \quad \varphi = \sin m(x - \alpha), \quad q \leqq m^2 - mp \operatorname{ctn} m(x - \alpha)$$

$$\left(\alpha < a < b < \alpha + \frac{\pi}{m}\right) \quad (m > 0).$$

$$(g)^* \quad \varphi = e^{-m\int p\,dx} \qquad q \leqq m(1-m)p^2 + mp' \text{ (provided } p \text{ has a continuous derivative).}$$

Formulæ of this sort might of course be multiplied indefinitely. To show how they are to be applied let us consider the simplest case of Bessel's equation

$$(9) \qquad y'' + \frac{1}{x}y' + y = 0.$$

Since p is here discontinuous at the point $x = 0$, we can consider only intervals which do not include, or even reach up to this point. Since the equation is unchanged by replacing x by $-x$, it will be sufficient to consider intervals in which x is positive.

Formulæ (a), (c), (g) yield us no information whatever with regard to this equation. Formula (b) is most serviceable here if we let $m = -1$. It then shows us that (9) is non-oscillatory in the interval $a \leqq x \leqq \frac{1}{2}$, where a is any small positive quantity. If we let $m = 2$ in formulæ (d) and (e), they each show us that (9) is non-oscillatory in the interval $a \leqq x \leqq b$, when $0 < a < b < 2$. This is the best result these two formulæ can be made to yield if we wish to consider an interval starting from a point arbitrarily near the point $x = 0$. Of the seven formulæ written above, (f) gives the best result when applied to intervals of the sort just described, since when we let $m = \frac{1}{2}\sqrt{2}$, $\alpha = -\frac{\pi}{2m}$, it shows us that (9) is non-oscillatory in the interval $a \leqq x \leqq b$ when

$$0 < a < b < \frac{\pi\sqrt{2}}{2} = 2.22.$$

This result, as we shall see in a moment, is nearly as good as any method could give us.

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* The special case $m = 1$ of this formula is noteworthy for the particularly simple result ($q \leqq p'$) which it yields; while the special case $m = \frac{1}{2}$ gives us the result we should have obtained by reducing (1) to the binomial form by a change of dependent variable only.

If on the other hand we wished to consider, for this same equation (9), intervals lying at a great distance from the origin, none of the formulæ above yield good results, the best being again (f), which when $m = 1$ shows us that (9) is non-oscillatory in any interval of length $\pi/2$. A much better result may be obtained by letting

$$\varphi = \frac{1}{\sqrt{x}} \sin m(x - a) \qquad \left(a < a < b < a + \frac{\pi}{m}\right).$$

This function leads us to the inequality

$$1 \leqq m^2 - \frac{3m - 2}{4x^2},$$

which just fails to be satisfied for large values of x when $m = 1$. We thus see that if l is any positive constant less than π, a positive constant M exists such that, in every interval of length l throughout which $x > M$, the equation (9) is non-oscillatory. That the function we have just used should give us an interval which is nearly twice as long as that given by formula (f) is the more remarkable because when x is large these two functions φ are, throughout an interval of length π, very nearly proportional to each other. If they were exactly proportional they would obviously lead to the same result.

We now leave these illustrative applications to equation (9).

Although we originally deduced condition (8) and its special cases as an extension, to the general equation (1), of the condition $q \leqq 0$ which we had established for the binomial equation (2), it turns out that some of these conditions give results for equation (2) which go beyond the result from which we started. Thus if we apply (f) to the special case $p = 0$ we get the important theorem

If throughout an interval of length less than l

$$q \leqq \frac{\pi^2}{l^2},$$

the equation (2) *is non-oscillatory in this interval.*[*]

Up to this point we have obtained merely sufficient conditions that a differential equation should be non-oscillatory.

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[*] Cf. the proof here given of this familiar theorem of Sturm with that given by Picard (l. c.).

A condition which turns out to be also necessary is obtained by taking as the function φ a solution of (1):

A necessary and sufficient condition that (1) *should be non-oscillatory in the interval* $a \leqq x \leqq b$ *is that it should have a solution which does not vanish in this interval.*

That this is a sufficient condition* is seen at once from the fact that either this solution or its negative satisfies conditions (7) and (8). To prove that it is also a necessary condition, assume that the interval $a \leqq x \leqq b$ is non-oscillatory, and consider the two solutions y_1 and y_2 of (1) which satisfy the conditions

$$y_1(a) = 0, \quad y_1'(a) > 0\,; \quad y_2(b) = 0, \quad y_2'(b) < 0.$$

Since neither of these solutions can vanish again in the interval, and since they are positive in the neighborhood of a and b respectively, they must be positive throughout the remainder of the interval. Accordingly $y_1 + y_2$ is a solution of (1) which is positive throughout the whole interval.

From the theorem just proved follows immediately this result:

A necessary and sufficient condition that the equation (1) is non-oscillatory in the interval $a \leqq x \leqq b$ *is that the solution of (1) which vanishes at a (or, if we prefer, at b), but is not identically zero, does not vanish again in the interval.*

This theorem gives us what is theoretically a perfect test; we have merely actually to compute the solution of (1) which vanishes at a, in the form of a series say, and to see whether or not this solution vanishes again in the interval. The difficulties involved in the computation may of course be so great in any special case as to make this method practically useless.

This last theorem if applied to (9) shows us that this equation is non-oscillatory in the interval $a \leqq x \leqq b$, where a is any small positive quantity and b is the smallest positive root of the Bessel function $J_0(x)$, viz. $b = 2.40 \cdots$.†

The condition (8) can also be stated in the following form which again gives us a necessary as well as a sufficient condition:

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* This also follows at once from the well known theorem of Sturm: *Between two successive roots of a solution of* (1) *lies one and only one root of any linearly independent solution;* and conversely this theorem follows from the theorem of the text

† This is actually the largest value that can be given to b, since every other solution of (9) has a root smaller than this. The proof of this fact is complicated by the presence of a singular point of (9) at $x = 0$. See, however, BULLETIN, March, 1897, p. 211.

If r *is a real function of* x *which, throughout the interval* $a \leqq x \leqq b$, *is continuous, and satisfies the condition* $r \leqq 0$, *then a necessary and sufficient condition that (1) be non-oscillatory in this interval is that the non-homogeneous equation*

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r \tag{10}$$

have a solution y *satisfying the relation*

$$y > 0 \quad (a \leqq x \leqq b).\text{*}$$

That this is really a necessary condition is seen from the fact that if (1) is non-oscillatory it has a solution positive throughout the interval, and by adding a sufficiently large positive multiple of this solution of (1) to an arbitrarily chosen solution of (10) we get the positive solution of (10) desired.

Another very important form into which condition (8) can be thrown is the following:

The function q_1 *being continuous in the interval* $a \leqq x \leqq b$ *and satisfying the condition*

$$q_1 \geqq q \qquad (a \leqq x \leqq b),$$

the equation (1) *will be non-oscillatory in this interval if the equation*

$$y'' + py' + q_1y = 0 \tag{11}$$

is non-oscillatory there.

For if (11) is non-oscillatory it has a solution φ positive throughout the interval $a \leqq x \leqq b$. Substituting φ in the first member of (1) gives us, when we take account of the fact that φ satisfies (11),

$$\varphi'' + p\varphi' + q\varphi = (q - q_1)\varphi \leqq 0 \qquad (a \leqq x \leqq b).$$

Thus φ satisfies conditions (7) and (8), and therefore (1) is non-oscillatory.†

In conclusion I will mention that condition (7) may be replaced by the somewhat less restrictive condition

$$\varphi > 0 \qquad (a < x < b), \tag{7'}$$

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* This theorem may also be proved directly and the other theorems of this paper deduced from it.

† This theorem may be proved directly by means of the methods used by Sturm (*Liouville's Journal*, vol. 1, p. 106) and is in fact a special case of one of Sturm's theorems. From it may be deduced the other theorems of this paper.

i. e., the case in which the function φ vanishes at one or both ends of the interval need not be excluded. The interval on the t-axis would, however, then extend to infinity in one or both directions, and the fundamental theorem concerning equation (2) from which we started would on longer be sufficient, but would have to be replaced by a theorem which states that, if $q \leqq 0$, no solution of (2) which vanishes at a finite point can approach a finite limit as x becomes either positively or negatively infinite, and that no solution of (2) can approach finite limits both when $x = +\infty$ and when $x = -\infty$.

The extension which our other theorems gain by the use of (7′) in place of (7) is easily seen. In using functions φ which vanish at one of the ends of the interval it is useful to know that if φ' also vanishes then φ cannot possibly satisfy (8),—a fact whose proof we also omit.

GÖTTINGEN, GERMANY,
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CONCERNING REAL AND COMPLEX CONTINUOUS GROUPS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, February 23, 1901.)

1. THIS paper aims to illustrate certain differences and certain analogies between related real and complex continuous groups. Lie's theory has been developed chiefly for the latter groups, the modifications necessary for real groups being treated quite briefly.

In §§ 2–4 are exhibited a real group in m variables and a real group in $2m$ variables, each of m^2 parameters, such that the corresponding complex groups are of like structure. In §§ 5–8, it is shown for $m = 2$ that the two real groups have different structures. Of the three proofs given, the first two are analytic and involve little technical knowledge of group theory, while the third group is geometric and gives a better insight into the nature of the question.

In § 10, it is illustrated for the case $m = 2$ how the general m-ary linear homogeneous complex continuous group gives rise to an isomorphic $2m$-ary linear homogeneous real continuous group. Similarly, the complex projective groups lead to groups of birational quadratic transformations.

The investigation has direct contact with the author's determination * of the structure of the largest group in the $GF[p^{2n}]$ leaving invariant $\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \cdots + \xi_m\bar{\xi}_m$, where $\bar{\xi}_i$ is conjugate to ξ_i with respect to the $GF[p^n]$; also with the paper by Moore † on the universal invariant of finite groups of linear substitutions.

2. Consider the group G_m of all substitutions

$$S: \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (i = 1, \cdots, m),$$

the coefficients and variables being complex numbers, such that S leaves formally invariant the Hermitian form

$$\Phi \equiv \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \cdots + \xi_m\bar{\xi}_m.$$

The conditions upon the coefficients are seen to be

$$(1) \qquad \sum_{i=1}^{m} a_{ij}\bar{a}_{ij} = 1, \quad \sum_{i=1}^{m} a_{ij}\bar{a}_{ik} = 0 \qquad (j, k = 1, \cdots, m; j \neq k).$$

It follows that the inverse of S has the form

$$S^{-1}: \qquad \xi_i' = \sum_{j=1}^{m} \bar{a}_{ji}\xi_j \qquad (i = 1, \cdots, m).$$

The group G_m is evidently continuous. To obtain the general infinitesimal transformation, set $I^2 = -1$ and

$$a_{ii} = 1 + (a_{ii} + Ib_{ii})\,\delta t, \quad a_{ij} = (a_{ij} + Ib_{ij})\,\delta t$$

$$(i, j = 1, \cdots, m; j \neq i).$$

Substituting these values in the relations (1) and retaining only the first power of δt, we find that

$$1 + 2a_{jj}\delta t = 1, \quad (a_{jk} + a_{kj}) + I(b_{kj} - b_{jk}) = 0$$
$$(j, k = 1, \cdots, m; j \neq k).$$

The conditions upon the general infinitesimal transformation

$$(2) \qquad \delta\xi_i \equiv \xi_i' - \xi_i = \sum_{j=1}^{m} (a_{ij} + Ib_{ij})\,\delta t \cdot \xi_j$$

are therefore the following

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* *Math. Annalen*, vol. 52, pp. 561–581.

† *Math. Annalen*, vol. 50, pp. 213–219.

$$a_{jk} = -a_{kj}, \quad b_{jk} = b_{kj} \qquad (j, k = 1, \cdots, m). \tag{3}$$

The general infinitesimal transformation of G_m is therefore a linear combination with *real* constant coefficients of

$$\begin{gathered} B_{ii}' \equiv I\xi_i \frac{\partial f}{\partial \xi_i}, \\ B_{ij}' \equiv I\xi_j \frac{\partial f}{\partial \xi_i} + I\xi_i \frac{\partial f}{\partial \xi_j}, \quad A_{ij}' \equiv \xi_j \frac{\partial f}{\partial \xi_i} - \xi_i \frac{\partial f}{\partial \xi_j}. \end{gathered} \tag{4}$$

Here B_{ij}' was obtained from (3) by setting $b_{ij} = b_{ji} = 1$ and the remaining constants all zero; A_{ij}' by setting $a_{ij} = -a_{ji} = 1$ and the remaining coefficients all zero.

The number of linearly independent transformations (4) is evidently m^2. If *complex* multipliers were allowed, we could derive from (4) the m^2 transformations

$$\xi_j \frac{\partial f}{\partial \xi_i} \qquad (i, j = 1, \cdots, m),$$

and therefore the general transformation of the m-ary linear homogeneous continuous group.

3. We obtain a continuous group R_{2m} on $2m$ real variables with real coefficients by replacing ξ_i by $X_i + IY_i$ for $i = 1, \cdots, m$ and separating reals and pure imaginaries. Relation (2) gives

$$\begin{aligned} X_i' + IY_i' - X_i - IY_i = \sum_{j=1}^{m} \{ (a_{ij}X_j - b_{ij}Y_j) \\ + I(a_{ij}Y_j + b_{ij}X_j) \} \delta t. \end{aligned}$$

Hence the general infinitesimal transformation of R_{2m} is

$$\delta X_i = \sum_{j=1}^{m} (a_{ij}X_j - b_{ij}Y_j)\delta t, \quad \delta Y_i = \sum_{j=1}^{m} (b_{ij}X_j + a_{ij}Y_j) \tag{5}$$
$$(i = 1, \cdots, m).$$

Denote by B_{ij} the transformation obtained by setting $b_{ij} = b_{ji} = 1$ and the remaining coefficients equal to zero; by A_{ij} that obtained by setting $a_{ij} = -a_{ji} = 1$ and the remaining coefficients equal to zero. Employing the usual abbreviations $p_i \equiv \frac{\partial f}{\partial X_i}$, $q_i \equiv \frac{\partial f}{\partial Y_i}$, we have

$$\begin{aligned} B_{ii} \equiv X_i q_i - Y_i p_i, \quad B_{ij} &\equiv X_j q_i - Y_j p_i + X_i q_j - Y_i p_j, \\ A_{ij} &\equiv Y_j q_i + X_j p_i - Y_i q_j - X_i p_j. \end{aligned}$$

Since $B_{ij} \equiv B_{ji}$, $A_{ij} \equiv -A_{ji}$, there are exactly m^2 independent transformations, from which the general infinitesimal transformation of R_{2m} may be derived as a linear expression with real coefficients. In view of the identity

$$\Phi \equiv \sum_{i=1}^{m} \xi_i \bar{\xi}_i = \sum_{i=1}^{m} (X_i^2 + Y_i^2)$$

the group R_{2m} is an orthogonal group. As a check it may be verified that the transformations B_{ij}, A_{ij} leave Φ absolutely invariant.

4. The following commutator (Klammerausdruck) relations are readily formed :

$$(6)\quad \begin{array}{l} (B_{ii}B_{jj})=0,\ (B_{ii}B_{ij})=A_{ij},\ (B_{ii}A_{ij})=-B_{ij},\ (B_{ij}B_{jk})=A_{ik}, \\ (A_{ij}B_{jk})=B_{ik},\quad (A_{ij}A_{jk})=A_{ik},\quad (B_{ij}A_{ij})=2B_{ii}-2B_{jj}, \end{array}$$

for $i, j, k = 1, \cdots, m$, with i, j, k distinct. If both subscripts of one symbol be different from the subscripts of the other, their commutator is zero.

It is readily verified that the transformations B_{ij}', A_{ij}' of §2 satisfy the commutator relations (6). This property would be expected to follow from the connection between G_m and R_{2m}. We may conclude that the continuous group with complex coefficients which is generated by the transformations B_{ij}, A_{ij} is isomorphic with the continuous complex group generated by B_{ij}', A_{ij}'.

Denote by B_{ij}'' the symbol obtained upon dropping the factor I from the symbol B_{ij}'. In the domain of real numbers, the transformations B_{ij}'', A_{ij}' generate the continuous group G_m' of all real linear homogeneous transformations in m variables. The symbols B_{ij}'', A_{ij}' do not satisfy the commutator relations (6). It is shown in §§ 5–8 that there does not exist in the real group G_2' $(m = 2)$ a set of four independent infinitesimal transformations which satisfy the commutator relations (6), so that G_2' and R_4 are non-isomorphic real continuous groups of four parameters each.

5. For $m = 2$, the relations (6) are the following :

$$(B_{11}B_{22}) = 0, \quad (B_{11}B_{12}) = A_{12}, \quad (B_{11}A_{12}) = -B_{12},$$

$$(B_{22}B_{12}) = -A_{12}, \quad (B_{22}A_{12}) = B_{12}, \quad (B_{12}A_{12}) = 2B_{11} - 2B_{22}.$$

The first derived group is therefore the three-parameter group generated by A_{12}, B_{12}, $B_{11} - B_{22}$. The only (ausgezeichnete) transformation whose commutator with B_{11}, B_{22}, B_{12}, A_{12} is zero is seen to be $B_{11} + B_{22}$, aside from a constant factor. Hence, if R_4 be isomorphic with G_2', $B_{11} + B_{22}$ must correspond with $\xi_1 \frac{\partial f}{\partial \xi_1} + \xi_2 \frac{\partial f}{\partial \xi_2}$ and the above three-parameter group with the first derived group of G_2'. To normalize R_4, set

$$Z_1 \equiv \tfrac{1}{2}(B_{11} - B_{22}), \quad Z_2 \equiv \tfrac{1}{2}A_{12}, \quad Z_3 \equiv -\tfrac{1}{2}B_{12}, \quad Z_4 \equiv B_{11} + B_{22}.$$

The above commutator relations then give

$$(7) \qquad (Z_1Z_2) = Z_3, \quad (Z_2Z_3) = Z_1, \quad (Z_3Z_1) = Z_2,$$

$$(8) \qquad (Z_4Z_1) = 0, \quad (Z_4Z_2) = 0, \quad (Z_4Z_3) = 0.$$

The first derived group of G_2' is generated by

$$V_1 \equiv \xi_1 \frac{\partial f}{\partial \xi_2}, \quad V_2 = \xi_1 \frac{\partial f}{\partial \xi_1} - \xi_2 \frac{\partial f}{\partial \xi_2}, \quad V_3 \equiv \xi_2 \frac{\partial f}{\partial \xi_1},$$

subject to the commutator relations

$$(9) \quad (V_1V_2) = -2V_1, \quad (V_2V_3) = -2V_3, \quad (V_3V_1) = -V_2.$$

To establish the non-isomorphism of R_4 and G_2', it suffices to prove that their first derived groups are non-isomorphic when considered as *real* continuous groups.

6. The most natural method of proof consists in showing that it is impossible to determine linear combinations of V_1, V_2, V_3 with real constant coefficients

$$Z_i' \equiv a_iV_1 + b_iV_2 + c_iV_3 \qquad (i = 1, 2, 3),$$

of determinant $\Delta \equiv \sum a_1b_2c_3 \neq 0$, which satisfy relations (7). We observe that

$$(Z_1'Z_2') \equiv -2V_1(a_1b_2 - b_1a_2) + V_2(a_1c_2 - c_1a_2) - 2V_3(b_1c_2 - c_1b_2).$$

The conditions that the right member shall equal Z_3' are

$$(10) \quad a_1b_2 - b_1a_2 = -\tfrac{1}{2}a_3, \quad a_1c_2 - c_1a_2 = b_3, \quad b_1c_2 - c_1b_2 = -\tfrac{1}{2}c_3.$$

By advancing the subscripts of a_i, b_i, c_i, we obtain the conditions for the identities $(Z_2'Z_3') = Z_1'$, $(Z_3'Z_1') = Z_2'$

$$(11) \quad a_2b_3 - b_2a_3 = -\tfrac{1}{2}a_1, \quad a_2c_3 - c_2a_3 = b_1, \quad b_2c_3 - c_2b_3 = -\tfrac{1}{2}c_1,$$

$$(12) \quad a_3b_1 - b_3a_1 = -\tfrac{1}{2}a_2, \quad a_3c_1 - c_3a_1 = b_2, \quad b_3c_1 - c_3b_1 = -\tfrac{1}{2}c_2.$$

In view of the relations (11) and (12),

$$-\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_3\,\Delta.$$

Applying the first relation (10), $\tfrac{1}{4}a_3 = a_3\Delta$. In a similar manner, or by advancing the subscripts (which does not alter Δ), we find

$$\tfrac{1}{4}a_1 = a_1\Delta, \quad \tfrac{1}{4}a_2 = a_2\Delta.$$

Since a_1, a_2, a_3 are not all zero, it follows that $\Delta = \frac{1}{4}$.

Multiplying equations (12) by $c_2, -b_2, a_2$ respectively and adding the resulting equations, we find

$$\Delta = -b_2^2 - a_2c_2. \tag{13}$$

Employing the multipliers $c_1, -b_1, a_1$, we find

$$0 = -b_1b_2 - \tfrac{1}{2}a_2c_1 - \tfrac{1}{2}a_1c_2. \tag{14}$$

In a similar manner, or by advancing the subscripts,

$$\Delta = -b_1^2 - a_1c_1. \tag{15}$$

By (14) and the second equation of set (10),

$$b_3^2 = (a_1c_2 - c_1a_2)^2 = 4b_1^2b_2^2 - 4a_1c_1a_2c_2.$$

Eliminating a_1c_1 and a_2c_2 by (15) and (13), and setting $\Delta = \frac{1}{4}$,

$$b_1^2 + b_2^2 + b_3^2 = -\tfrac{1}{4}.$$

But this equation is impossible for real values of the b_i.

7. A second proof is derived from the following investigation which gives certain interesting properties of the group Γ generated by Z_1, Z_2, Z_3, subject to the relations (7). Set

$$\begin{aligned} Z_1 &= \alpha_1U_1 + \alpha_2U_2 + \alpha_3U_3, \\ Z_2 &= \beta_1U_1 + \beta_2U_2 + \beta_3U_3, \\ Z_3 &= \gamma_1U_1 + \gamma_2U_2 + \gamma_3U_3, \end{aligned} \qquad \Delta \equiv \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0.$$

We obtain by solution the most general set of independent infinitesimal transformations U_1, U_2, U_3, of the group Γ. We seek the commutator relations of the U_i. Denote by α_i' the first minor (without prefixed sign) of α_i in Δ, β_i' the first minor of β_i, γ_i' the first minor of γ_i. Form (Z_2Z_3) and equate the result to Z_1; expand similarly $(Z_3Z_1) = Z_2$, $(Z_1Z_2) = Z_3$. The results are

$$\begin{aligned} Z_1 &= \alpha_1'(U_2U_3) - \alpha_2'(U_3U_1) + \alpha_3'(U_1U_2), \\ Z_2 &= -\beta_1'(U_2U_3) + \beta_2'(U_3U_1) - \beta_3'(U_1U_2), \\ Z_3 &= \gamma_1'(U_2U_3) - \gamma_2'(U_3U_1) + \gamma_3'(U_1U_2). \end{aligned}$$

The determinant of the coefficients equals Δ^2, being equal to the determinant of the first minors of Δ. Moreover,

$$\begin{vmatrix} \beta_2' & \beta_3' \\ \gamma_2' & \gamma_3' \end{vmatrix} = \alpha_1\Delta.$$

The solution of the above relations therefore gives

$$\begin{aligned}\Delta(U_2U_3) &= \alpha_1Z_1 + \beta_1Z_2 + \gamma_1Z_3,\\ \Delta(U_3U_1) &= \alpha_2Z_1 + \beta_2Z_2 + \gamma_2Z_3,\\ \Delta(U_1U_2) &= \alpha_3Z_1 + \beta_3Z_2 + \gamma_3Z_3.\end{aligned}$$

The matrix of the coefficients on the right is the transposed of the matrix of the coefficients of the U_i in the expressions for the Z_i. Eliminating the Z_i, we have

$$\begin{array}{r|lll} & U_1 & U_2 & U_3 \\ \hline \Delta(U_2U_3)= & \alpha_1^2+\beta_1^2+\gamma_1^2 & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 & \alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3 \\ \Delta(U_3U_1)= & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 & \alpha_2^2+\beta_2^2+\gamma_2^2 & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_2 \\ \Delta(U_1U_2)= & \alpha_1\alpha_3+\beta_1\beta+\gamma_1\gamma_3 & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3 & \alpha_3^2+\beta_3^2+\gamma_3^2 \end{array}$$

The symmetry of the matrix of coefficients is in accord with a known property of the group.*

In order that the transformations U_i should satisfy the same commutator relations (9) as the transformations V_i it is necessary that $(U_2U_3) = -2U_3$, so that $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 0$. For real values of α_1, β_1, γ_1, this requires $\alpha_1 = \beta_1 = \gamma_1 = 0$, contrary to hypothesis. Hence the real group Γ of the Z_i is not isomorphic with the real group of the V_i.

To obtain the most general set of three infinitesimal transformations U_i of Γ which satisfy the same commutator relations (7) as the transformations Z_i themselves,

$$(U_2U_3) = U_1, \quad (U_3U_1) = U_2, \quad (U_1U_2) = U_3,$$

it is necessary and sufficient to take solutions α_i, β_i, γ_i of

$$\Delta = \alpha_1^2 + \beta_1^2 + \gamma_1^2, \quad \Delta = \alpha_2^2 + \beta_2^2 + \gamma_2^2, \quad \Delta = \alpha_3^2 + \beta_3^2 + \gamma_3^2.$$

$$0 = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, \quad 0 = \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3,$$

$$0 = \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3.$$

These are the conditions for an orthogonal substitution, the invariant relation being

$$Z_1^2 + Z_2^2 + Z_3^2 \equiv \Delta\,(U_1^2 + U_2^2 + U_3^2).$$

8. To give a third proof, based upon geometric considerations, it suffices to consider the adjoint groups of the three-parameter groups in question. The adjoint of the group of the V_i is

$$2e_2\frac{\partial f}{\partial e_1} - e_3\frac{\partial f}{\partial e_2}, \quad -2e_1\frac{\partial f}{\partial e_1} + 2e_3\frac{\partial f}{\partial e_3}, \quad e_1\frac{\partial f}{\partial e_2} - 2e_2\frac{\partial f}{\partial e_3},$$

having as its only invariant curve the real conic

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*Lie-Scheffers, Vorlesungen uber continuierliche Gruppen, p. 567.

$$(16) \qquad e_1 e_3 + e_2^2 = 0.$$

The adjoint group of the group of the Z_i is

$$e_3 \frac{\partial f}{\partial e_2} - e_2 \frac{\partial f}{\partial e_3}, \quad -e_3 \frac{\partial f}{\partial e_1} + e_1 \frac{\partial f}{\partial e_3}, \quad e_2 \frac{\partial f}{\partial e_1} - e_1 \frac{\partial f}{\partial e_2},$$

having as its only invariant the imaginary conic

$$(17) \qquad e_1^2 + e_2^2 + e_3^2 = 0.$$

Now the replacement of one complete set of independent infinitesimal transformations of a group by a second complete set merely gives rise to a linear homogeneous transformation upon the variables e_i of the adjoint group. The latter will be a real transformation if the second set is expressed in terms of the first by real coefficients. Since the equations (16) and (17) can not be transformed into each other by a real ternary substitution, it follows that the transformations V_i are not expressible as real linear functions of the Z_i.

The method of reduction of three-parameter non-integrable groups to a normal type given in Lie-Scheffers, Vorlesungen, pp. 566–568 is immediately applicable only to complex groups. For real groups there are two (and, indeed, only two*) distinct cases, according as the invariant conic (necessarily non-degenerate) is real or imaginary. The two methods there given as optional for complex groups are to be differentiated for real groups to correspond to the cases of real and imaginary conics, yielding respectively the normal types (I) and (I′) of p. 568, or types (9) and (7) respectively of this paper.

9. The real four-parameter groups G_2' and R_4 have been proved to have different structures. Applying the imaginary transformation of variables

$$X_1 = x_1, \quad Y_1 = y_1, \quad X_2 = Ix_2, \quad Y_2 = Iy_2 \qquad (I^2 = -1),$$

the infinitesimal transformations B_{11}, B_{22}, B_{12}, A_{12} of R_4 become

$$b_{11} \equiv x_1 \frac{\partial f}{\partial y_1} - y_1 \frac{\partial f}{\partial x_1}, \quad b_{22} \equiv x_2 \frac{\partial f}{\partial y_2} - y_2 \frac{\partial f}{\partial x_2},$$

$$b_{12} \equiv x_2 \frac{\partial f}{\partial y_1} - y_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial y_2} + y_1 \frac{\partial f}{\partial x_2},$$

$$a_{12} \equiv y_2 \frac{\partial f}{\partial y_1} + x_2 \frac{\partial f}{\partial x_1} + y_1 \frac{\partial f}{\partial y_2} + x_1 \frac{\partial f}{\partial x_2}.$$

---

*An irreducible ternary quadratic form with real coefficients is reducible either to $b(e_1^2 + e_2^2 + e_3^2)$ or to $b(e_1^2 + e_2^2 - e_3^2)$.

They satisfy the commutator relations

$$\begin{aligned}&(b_{11}b_{22}) = 0, \qquad (b_{11}b_{12}) = a_{12},\ (b_{11}a_{12}) = -b_{12},\\&(b_{22}b_{12}) = -a_{12},\ (b_{22}a_{12}) = b_{12},\ (b_{12}a_{12}) = -2b_{11} + 2b_{22}.\end{aligned}$$

Except for the last relation, these are identical with the commutator relations of B_{11}, B_{12}, B_{22}, A_{12} (§5). Setting

$$W_1 \equiv a_{12},\ W_2 \equiv -b_{12},\ W_3 \equiv b_{11} - b_{22},\ W_4 \equiv b_{11} + b_{22},$$

we have the commutator relations

$$\begin{aligned}&(W_1W_2) = -2W_3,\ (W_2W_3) = 2W_1,\ (W_3W_1) = 2W_2,\\&(W_4W_1) = 0, \qquad (W_4W_2) = 0, \qquad (W_4W_3) = 0.\end{aligned}$$

These relations are also satisfied by the transformations

$$w_1 \equiv xq + yp,\ w_2 \equiv xp - yq,\ w_3 \equiv xq - yp,\ w_4 \equiv xp + yq,$$

which generate the general binary group G_2'. *By an imaginary transformation of variables, R_4 may be given a real form having the same structure as G_2'.*

10. Consider the group G of all binary transformations

$$S: \quad \begin{cases} X' = \alpha X + \gamma Y \\ Y' = \beta X + \delta Y \end{cases} \qquad (\alpha\delta - \beta\gamma = 1)$$

upon complex variables X, Y with complex coefficients of determinant unity. Let $I^2 = -1$ and set

$$X = x + Ix_1,\ Y = y + Iy_1,\ \alpha = a + Ia_1,\ \beta = b + Ib_1,\ \text{etc.}$$

Then S corresponds to the quaternary transformation

$$\Sigma: \quad \begin{array}{r|cccc} & x & x_1 & y & y_1 \\ \hline x' = & a & -a_1 & c & -c_1 \\ x_1' = & a_1 & a & c_1 & c \\ y' = & b & -b_1 & d & -d_1 \\ y_1' = & b_1 & b & d_1 & d \end{array}$$

The relation $\alpha\delta - \beta\gamma = 1$ gives

$$ad - bc - a_1d_1 + b_1c_1 = 1, \quad ad_1 + a_1d - bc_1 - b_1c = 0.$$

The determinant of Σ is seen to equal

$$(ad - bc - a_1d_1 + b_1c_1)^2 + (ad_1 + a_1d - bc_1 - b_1c)^2 = 1.$$

To the product S_1S_2 of two substitutions of the form S, corresponds the product $\Sigma_1\Sigma_2$ of the corresponding substitutions Σ. Hence the group G is isomorphic with the group Γ of the substitution $\sum$. But $\sum$ reduces to the identity only when S is the identity. Hence the isomorphism is holoedric.

By the usual method, the general infinitesimal transformation of Γ is found to be a linear combination of the following linearly independent transformations :

$$\begin{array}{c|cccc}
 & \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial y_1} \\
\hline
A & x & x_1 & -y & -y_1 \\
A_1 & -x_1 & x & y_1 & -y \\
B & 0 & 0 & x & x_1 \\
B_1 & 0 & 0 & -x_1 & x \\
C & y & y_1 & 0 & 0 \\
C_1 & -y_1 & y & 0 & 0
\end{array}$$

The real group Γ therefore possesses no invariant. The non-vanishing second minors of the matrix of coefficients are

$$P \equiv x^2 + x_1^2, \quad Q \equiv y^2 + y_1^2, \quad R = xy + x_1y_1, \quad S = x_1y - xy_1.$$

Upon them the group Γ gives rise to the following transformations :

$$\begin{array}{c|cccc}
 & \dfrac{\partial f}{\partial P} & \dfrac{\partial f}{\partial Q} & \dfrac{\partial f}{\partial R} & \dfrac{\partial f}{\partial S} \\
\hline
A & 2P & -2Q & 0 & 0 \\
A_1 & 0 & 0 & -2S & 2R \\
B & 0 & 2R & P & 0 \\
B_1 & 0 & -2S & 0 & -P \\
C & 2R & 0 & Q & 0 \\
C_1 & 2S & 0 & 0 & Q
\end{array}$$

The determinants of the fourth order of this matrix are all identically zero. To obtain the homogeneous invariants, we annex Euler's homogeneous operator

$$H \equiv P\frac{\partial f}{\partial P} + Q\frac{\partial f}{\partial Q} + R\frac{\partial f}{\partial R} + S\frac{\partial f}{\partial S}.$$

The determinant of the coefficients of A, A_1, B, H equals $8PR(S^2+R^2-PQ)$. The determinant of A, A_1, B_1, H equals $-8PS(S^2+R^2-PQ)$. In this way the only homogeneous invariant is seen to be

$$\psi \equiv S^2 + R^2 - PQ.$$

In terms of the initial variables x, x_1, y, y_1, we see that ψ vanishes identically. Also

$$P \equiv | x + Ix_1 | \equiv | X |, \quad Q \equiv | Y |,$$

$$\frac{Y}{X} \equiv \frac{R - IS}{P}, \quad \frac{X}{Y} \equiv \frac{R + IS}{Q}.$$

The group G is hemiedrically isomorphic with the group of linear fractional substitutions

$$(18) \qquad Z' = \frac{\alpha + \gamma Z}{\beta + \delta Z}, \quad Z \equiv \frac{Y}{X}.$$

The quaternary group on P, Q, R, S is isomorphic with a ternary fractional group on Q/P, R/P, S/P. But

$$\frac{Q}{P} \equiv \left(\frac{R}{P}\right)^2 + \left(\frac{S}{P}\right)^2.$$

Eliminating Q/P, we obtain a group of birational quadratic transformations in the plane. It may evidently be obtained more directly from the transformations (18).

THE UNIVERSITY OF CHICAGO,
January, 1901.

ON HOLOMORPHISMS AND PRIMITIVE ROOTS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, February 23, 1901.)

IN an earlier note* it was observed that every holomorphism of an abelian group with itself can be obtained by establishing an isomorphism between the abelian group and one of its subgroups (which may sometimes be the entire group) and associating the product of corresponding operators with the original operator of the group. The present note is devoted to some additional developments along this line and especially to some elementary results in the theory of numbers which may be derived by this method.

Let s_1 represent an operator of order p^m (p being any prime number) and let P, the group generated by s_1, be made isomorphic with one of its subgroups of order p^{m_1}, $m_1 < m$. Each operator of P is transformed, by some operator t in the group of isomorphisms of P, into itself multiplied by the corresponding operator in this isomorphism. Assuming that $t^{-1} s_a t = s_{a+1} s_a$, we have

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* BULLETIN, Vol. 6 (1900), p. 337.

$$t^{-n} s_a t^n = s_{a+n} s_{a+n-1}^{n} \cdots s_{a+n-r}^{\frac{n(n-1)\cdots(n-r+1)}{r!}} \cdots s_{a+1}^{n} s_a.$$

It is easy to prove that the order of the product of the operators which are multiplied into s_a is equal to the order of s_{a+1}^n whenever p is odd. In case n is prime to p this follows directly from the fact that each of the factors which precede s_{a+1} is of a lower order than s_{a+1}. In general, let $n = kp\lambda$, k being prime to p. The exponent of $s_{a+\beta}$ is divisible by $p^{\lambda-m'}$, where m' is the exponent of the highest power of p that is contained in β, since the product of n successive numbers is divisible by $n\,!$. As the order of $s_{a+\beta}$ does not exceed the order of $(s_{a+1})^{p^{\beta-1}}$, the order of the power of $s_{a+\beta}$ which occurs in the above formula cannot exceed $(s_{a+1})^{p^{\lambda+\beta-m'-1}}$. Hence it is less than the order of s_{a+1}^n whenever $\beta > 1$, and *the product of all the factors which are multiplied into s_a is of the same order as s_{a+1}^n when p is odd.*

When p is even we assume that $m_1 < m - 1$. With the same notation as above it is clear that the order of $s_{a+\beta}$ does not exceed the order of $(s_{a+1})^{2^{2(\beta-1)}}$. Hence the order of the power of $s_{a+\beta}$ in the formula cannot exceed

$$(s_{a+1})^{2^{\lambda+2\beta-m'-2}}$$

As $m' + 2$ is less than 2β whenever $\beta > 1$, it follows that in this case the order of the product of the factors which are multiplied into s_a is again equal to s_{a+1}^n. Hence *t is always of order p^{m_1} and the group of isomorphisms of P contains a cyclic subgroup of order p^{m-1} when p is odd and one of order 2^{m-2} when p is even.*

The group of isomorphisms of a cyclic group is abelian,* and can be represented as a regular substitution group whose elements correspond to the operators of highest order in the cyclic group.† Hence the group of isomorphisms of the cyclic group of order p^m is of order $p^{m-1}(p-1)$. In particular, the group of isomorphisms I of the cyclic group of order 2^m is of order 2^{m-1}. We have just found that I contains a cyclic subgroup of order 2^{m-l}, formed by all its operators which transform into itself an operator of order 2^l $(l > 1)$ in P.

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* *Trans Amer. Math. Soc.*, vol. 1 (1900), p. 397.

† The order of a cyclic group is said to have primitive roots whenever its group of isomorphisms is cyclic.

The group I contains an operator s' of order two which transforms each operator of P into its inverse. As s' is not contained in the above cyclic subgroup of order 2^{m-2}, which is composed of all the operators of I which transform an operator of order four in P into itself, it and this subgroup must generate I.

It is now easy to determine the exponent to which a number belongs mod 2^m, since this exponent is the order of the corresponding operator in I. In the above mentioned cyclic subgroup of order 2^{m-2}, an operator of order 2^k is commutative with the operators of order 2^{m-k} in P, but not with those of order 2^{m-k+1}. From this fact and the fact that s' transforms each operator of P into its inverse it follows that all the numbers which belong to exponent 2^k $(k > 1)$ mod 2^m are of the form $\pm (l\,2^{m-k} + 1)$ where l is any one of the $\varphi(2^k)$ numbers not greater than 2^k and prime to 2^k; and *vice versa*. When $k = 1$ we have to add $2^m - 1$ to the numbers obtained in this way. Hence the numbers which belong to exponent 2^{m-2} are $\equiv 3$ or 5 mod 8.*

From what is proved above it follows that the group of isomorphisms I_1 of P contains a cyclic subgroup of order $p^{m'-1}$, p being any odd prime, which is composed of all the operators of I_1 commutative with each of the operators of order p in P. By adding to this subgroup the operators which transform transitively these $p - 1$ operators of order p, we obtain the $p^{m-1}(p-1)$ operators of I. The group of order $p - 1$ according to which the operators of order p are transformed contains no more than d operators whose orders divide d, any factor of $p - 1$, since $x^d \equiv 1$ mod p can have no more than d roots † Hence it cannot have two subgroups of the same order and must therefore be cyclic. Since I_1 is abelian its operators of highest order are obtained by multiplying the operators of order $p - 1$ in this cyclic subgroup by the operators of order p^{m-1} in the above mentioned cyclic subgroup of order p^{m-1}. Hence I_1 is cyclic and the primitive roots of p^m are also primitive roots of p.‡ It may be observed that *the above furnishes an independent proof of the existence of primitive roots of* p^m. That the cyclic group of order $2p^m$ has the same group of isomorphisms as the cyclic group of order p^m follows directly from the fact that the operator of order two in the former must correspond to itself in every holomorphism of the group with itself. Hence $2p^m$ also has primitive roots.

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* Cf. Mathews, Theory of numbers, 1892, p. 30.

† Gauss, Disquisitiones Arithmeticæ, 1801, Art 54.

‡ Ibid , Art. 92.

The holomorphisms mentioned in the second and third paragraphs show that all numbers of the form $kp^{m-a}+1$, where k is any one of the $\varphi(p^a)$ natural numbers which are not greater than p^a and prime to p^a, belong to the exponent p^a; a being any positive integer less than m when p is odd and less than $m-1$ when p is even. In the preceding paragraph it is proved that these are the only numbers which belong to an exponent which is a power of an odd prime. The product of all the numbers which belong to exponent p^{m-1}, and the $\varphi(p-1)$ powers of a number which is not a primitive root of p^m, but belongs to exponent $p-1 \bmod p$, will clearly give all the primitive roots of p^m, since the corresponding operators in I_1 are all its operators of order $p^{m-1}(p-1)$. In particular the primitive roots of 3^m are the products of 3^m-1 and the numbers of the form $3l+1$, l being any one of the positive integers not greater than 3^{m-1} and prime to 3^{m-1}.

The necessary and sufficient condition that an operator of I_1 corresponds to a primitive root of p^a is that its order is divisible by $p-1$ and that its $(p-1)$th power corresponds to a holomorphism of P with itself which may be obtained by establishing a $p, 1$ isomorphism between P and its subgroup of order p^{m-1}. Hence *the primitive roots of* p^a $(a>1)$ *are also the primitive roots of every power of* p.* The pth power of a primitive root of p is also a primitive root of p, but the pth power of a primitive root of p^a is not a primitive root of p^a. The primitive roots of p are therefore not always primitive roots of p^a. In fact, we observe directly, from the orders of the operators of I_1, that just $\frac{p-1}{p}$ of the primitive roots of p which are less than p^m are also primitive roots of p^m.

The preceding considerations can readily be applied to the general cyclic group C of order $2^{a_0}p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$ $(p_1, p_2, \cdots, p_m$ being odd prime numbers). By making C isomorphic with its subgroup of order $2^{a_0'}p_1^{a_1'}p_2^{a_2'}\cdots p_m^{a_m'}$ (where $a_0' < a_0 - 1$; $a_\gamma' < a_\gamma$, $\gamma = 1, 2, \cdots, m$), and multiplying the corresponding operators, we obtain a holomorphism of C with itself, which corresponds to an operator of order $2^{a_0'}p_1^{a_1'}p_2^{a_2'}\cdots p^{a_m'}$ in its group of isomorphisms. Since the latter group is the direct product† of the groups of isomorphisms of the cyclic groups of orders $2^{a_1}, p_1^{a_1}, p_2^{a_2}, \cdots, p_m^{a_m}$ and since the group of isomorphisms of each one of these groups involves operators of order two whenever the order of the group exceeds two, the group of isomorphisms of C is cyclic only when $a_0 = 0$ or 1 and just one of the other exponents differs from 0, or when $a_0 = 1$ or 2 and all the other exponents are 0.*

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* Lebesgue, *Liouville's Journal*, vol. 19 (1854), p. 344.

† *Trans. Amer. Math. Soc.*, vol. 1 (1900), p. 396.

CORNELL UNIVERSITY,
February, 1901.

BESSEL FUNCTIONS.

Einleitung in die Theorie der Bessel'schen Funktionen. By PROFESSOR J. H. GRAF und DR. E. GUBLER. Zweites Heft: *Funktionen zweiter Art.* Bern, Wyss and Co., 1900.

THE first part of this work appeared in 1898 and was reviewed in the BULLETIN, February, 1899, pp. 253–8. The general arrangement of the second part is similar to that of the first, the authors again emphasizing the fact that the work is done in the spirit of Schläfli's lectures, the manuscripts of which were in their hands, though many problems are extended and modernized. This fact explains the absence of many important phases of the theory of the Bessel functions which one might expect in a symmetric treatise. Moreover, the authors have been rather overgenerous in their references to papers originating at Bern, omitting others which contained proofs of fundamental theorems prior to their discovery by the Bern school, although probably no plagiarism could be charged. Several fundamental theorems by American authors have received no recognition in the book.

Here, as in Volume I, the loop integral is the principal factor in the investigation, and next in importance is the expansion in series. The differential equation is less frequently used. The procedure is rather original, and frequently markedly different proofs for well-known theorems are given, which in some instances have led to detection of error in papers already published.

The only attempt at a concrete illustration or application is the expansion of a few functions in terms of Bessel functions, though the relations which exist between these functions and others are quite fully brought out.

The second part begins with the expansion of $\frac{1}{x-y}$ in terms of Bessel functions, the result being

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* Gauss, Disquisitiones Arithmeticæ, 1801, Art. 92.

$$\frac{1}{x-y} = J_0(y)\cdot\frac{1}{x} + \sum_{n=1}^{\infty} J_n(y)\int_0^{\frac{N}{x}} e^{-xs}\left(t^n - \frac{(-1)^n}{t^n}\right)ds.*$$

$$[s=\tfrac{1}{2}(t-t^{-1})], \quad (|x|>|y|), \quad (\lim N=\infty).$$

The part under the integral sign is denoted by $2\,O_n(x)$, and $O_n(x)$ is called the Bessel function of the second kind. This terminology is unusual, since the differential equation for $J_a(x)$ is not satisfied by $O_a(x)$, but in other respects $O_a(x)$ is quite analogous to $J_a(x)$. The authors suggest the analogy between the Bessel functions of the first and second kind on the one hand and spherical harmonics of the first and second kind, as defined by Neumann, on the other.

The symbol n is used to denote an integral parameter; the form of the infinite series for $O_n(x)$ is then derived, and the numerical coefficients calculated for $n=1$ to $n=11$. This method is then compared with that of Neumann for obtaining equation (1). $O_n(x)$, $J_n(x)$ are both shown to exist in a Laurent ring. Any continuous and differentiable function can be expanded in but one way in terms of Bessel functions.† The discussion of integrals of products of J, O closes the chapter.

In the following chapter the related function $S_n(x)$ is introduced:

$$S_n(x) = \int_1^{\frac{N}{x}} e^{-xs}\left(t^n - \frac{(-1)^n}{t^n}\right)\frac{dt}{t},$$

$$\left(O_n(x) = \frac{\cos^2\frac{1}{2}n\pi}{x} + \frac{n}{2x}S_n(x)\right).$$

The expression $\cos^2\frac{1}{2}n\pi$, $=0$ or 1, $n=1 \bmod 2$, $0 \bmod 2$ causes some confusion, both in this and later chapters, but the difficulty is easily removed by changing a limit in a summation. The numerical calculation of $S_n(x)$ is given for $n=1$ to $n=12$. $S_n(x)$ is always a polynomial in x^{-1}.

$$\frac{\cos^2\frac{1}{2}\pi n}{x}$$

is eliminated and $S_n(x)$ expressed as a (finite) series in terms of $O_\lambda(x)$. The differential equation is found to be

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* The notation $\overset{a}{J}$ of the work reviewed is here replaced by J_a, etc.

† In my review of Part 1, I carelessly attributed this theorem to Schlömilch. This was simply an error of my own; the statement was not made in the book, but Schlömilch's name was used in a different connection.

$$x^2 \frac{d^2}{dx^2} S_n + x \frac{dS_n}{dx} + (x^2 - n^2) S_n = 2x \sin^2 \tfrac{1}{2} n\pi$$

$$+ 2n \cos^2 \tfrac{1}{2} n\pi = 2x \text{ or } 2n \text{ as } n \equiv 0,\ 1 \bmod 2\,;$$

that for O is

$$\square\, O_n + O_n = x \cos^2 \tfrac{1}{2} n\pi + n \sin^2 \tfrac{1}{2} n\pi,$$

$\square\, J_n(x) = 0$ being the differential equation for $J_n(x)$.

The chapter closes with the integration of this differential equation, which results in expressing $S_n(x)$ and $O_n(x)$ in terms of $J_n(x)$, $K_n(x)$.

Chapter VIII introduces two new functions, $T_n(x)$, $U_n(x)$, defined by series, somewhat analogous to the partial summations of $J(x)$ and of $K(x)$. They are next expressed as integrals,

$$T_n(x) = \frac{2}{\pi i} \int_0^\pi \left(\varphi - \frac{\pi}{2} \right) e^{-i(x \sin \phi - n\phi)}\, d\varphi.$$

In deriving the differential equation for $T_n(x)$, $\cos^2 \frac{1}{2} n\pi$ intrudes again and the authors have chosen an infelicitous expression for removing it. In fact, taking the statement pp. 50–52 literally, an actual error would be made. This is about the only difficult part to follow in the text.

The function T is expressed in terms of J_λ and also as a definite integral—a similar discussion follows for U. K, J, T, S are shown to satisfy the relation

$$(-1)^n y_{-n} = y_n.$$

At the end of the chapter the relation between these functions and the y of Neumann, of Hankel, and of Weber are given, the first being expressed by the equation

$$y_n(x) = \log x . J_n(x) - \tfrac{1}{2} S_n(x) + \tfrac{1}{2} T_n(x) - U_n(x),$$

and the others are also given in detail. The functions S_n, T_n, U_n, are called Schläfli functions.

Chapter X deals with the addition theorem; it is prefaced by a historical introduction which concisely gives the development of the problem. The method of proof is quite consequent: it consists in expressing $J_n(x + y)$ as a definite (loop) integral

$$J_n(x + y) = \frac{1}{2\pi i} \int e^{(x+y)s} t^{-n-1} dt, \qquad \left[s = \tfrac{1}{2}(t - t^{-1})\right],$$

$$e^{ys} = \sum_{\lambda = -\infty}^{\infty} J_\lambda(y) t^\lambda,$$

$$J_n(x+y) = \sum_{\lambda=-\infty} J\ (y) J_{n-\lambda}(x), \qquad (|y|<|x|)$$

By simple and natural transformations, the functions

$$J_n(x), \quad K_n(x), \quad O_n(x), \quad S_n(x), \quad T_n(x),$$

are shown to satisfy the same addition theorem, namely,

$$Z_n(x+y) = \sum_{\lambda=-\infty}^{\infty} Z_{n-\lambda}(x) J_\lambda(y) \, ;$$

similar expressions are derived for the argument $x-y$, for parameter n and $-n$. The chapter concludes with a similar discussion of the product of two Bessel functions. To me this chapter appears very successful in its consistent method of procedure

The next chapter deals with the expansion into a continued fraction of the ratio of two Bessel functions whose parameters differ by unity; it is preceded by a sketch of the historical development of the problem, a feature that is repeated in every subsequent chapter. Several pages are devoted to the expansion of particular functions.

The Schläfli function $P_m^{(a)}(x)$ is shown to be a polynomial of order m in x^{-1}. It is developed for values of m from -2 to 8 in terms of an arbitrary parameter a; then a recurring process gives the value of the function for other negative values of the integer m. The chapter closes with the application to some particular cases, $a = \frac{1}{2}$, $a = -\frac{1}{2}$, $\lim_{m=\infty} P_m^{(a)}(x)$.

Chapter XII treats of the classic problem of the relation of the Bessel function to the hypergeometric series. The treatment is quite original and direct, the method being somewhat independent of the older memoirs. The problem is to determine the value of the integral

$$S = \int_0^\infty J_a(x) e^{-bx} x^{c-1} dx \, ;$$

the result is

$$S = \frac{\Gamma(a+c)}{2^a \Gamma(a+1) b^{a+c}} F\left(\frac{a+c}{2}, \frac{a+c+1,}{2}, a+1, -\frac{1}{b^2}\right);$$

the conditions for convergence are carefully discussed.

An interesting case is discussed wherein $b = \pm i$; the region in which the function exists and the transformed path of the loop integral are well treated. A second proof is given, depending on a curvilinear integral; some new relations between F functions are incidentally found, among them being

$$F\left(\frac{a+\gamma-1}{2},\ \frac{a+\gamma}{2},\gamma,-\frac{4x(1-x)}{(1-2x)^2}\right)$$
$$=(1-2x)^{a+\gamma-1}(1-x)^{1-\gamma}F(a,1-a,\gamma,x).$$

This last equation is then developed directly by means of the definite integral; the path of integration is varied and the moduli of periodicity obtained by crossing the section are then discussed. Here again the special power of the curvilinear integral is exhibited, in the use of which the authors have shown considerable skill. The function considered is

$$T=\frac{t^2}{4(t-1)},$$

and the curve of section (Grenzscheide) is a limaçon. The chapter closes with a discussion of a few particular cases, including besides some well known results a few new ones regarding integrals of Bessel functions.

In the final chapter Weber's discontinuous integral

$$\int_0^\infty J_1(x)J_0(ax)dx$$

is discussed and generalized for any parameters; this is done by Dr. Gubler The method employed is an application of the principles established in the preceding chapter, with the selection of an appropriate path of integration for each case.

A short appendix is added, in which still another form of the integral is obtained, and another section reduces the differential equation of the Bessel function of the first kind to a Riccati equation; and gives the form of the solution as a function of two independent $J(x)$ functions.

The bibliography which introduces each chapter, and the list of sources quoted which is appended to each part, while not always complete and not always giving the original source, still form a valuable part of the book. This work and the treatise of Gray and Mathews supplement each other, each being somewhat one sided when studied alone.

A number of typographical errors already found are corrected in a list inserted in Part II; but a good many more are still in the text, though few, if any, would prove a source of annoyance to the reader.

Any worker in Bessel functions will find the work a helpful text.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
March 9, 1901.

SHORTER NOTICES.

Lesioni sulle Teoria delle Superficie. By GREGORIO RICCI. Padova, 1898.

THIS lithographed volume of about 400 pages gives the formal development and application to the theory of surfaces of the instrument of analysis termed Calcolo differenziale assoluto by the author. The method leads to formulæ and equations always presenting themselves under the same form for any system of independent variables, and the difficulties which are incidental and formal rather than intrinsic are thus to some extent done away with, and the research assumes a uniformity absent in other methods. The entire discussion is based on the properties of differential quadratic forms, and the Introduction, of about 130 pages, contains a rather complete exposition of the methods of the calculus with a more or less full treatment of differential quadratic forms in general and binary forms in particular. The novelty and generality of the method and notation require of the reader not a little effort; but, these difficulties of form once overcome, the various geometrical applications follow easily.

Chapter I is devoted to generalities on linear homogeneous partial differential equations of the first order and to complete systems. Chapters II and III treat differential quadratic forms and develop the absolute calculus for n variables. In chapters IV and V these forms are classified and differential invariants are formed, while chapter VI applies the calculus to two variables.

Part I, comprising the second third of the book, treats surfaces as flexible and inextensible, while Part II is taken up with the theory of surfaces considered as having a rigid form in space Chapter IV of this part treats special classes of surfaces, while chapter VI is devoted to surfaces of the second degree.

The treatise is by no means a complete exposition of the whole theory of surfaces, such subjects as infinitesmal deformations, pseudospherical geometry, and triply orthogonal systems being barely mentioned; but the idea of the volume seems to be to give some notion of the power and elegance of the absolute calculus. This has a wider application than that indicated here. many questions of pure mathematics and of mathematical physics being advantageously treated by it; and those accustomed to the classic method and notation, as found in Darboux and Bianchi for instance, will find the book well worth reading if for no other reason than to look at the theory from another and entirely different aspect.

G. O. JAMES.

Leçons sur la Théorie des Formes et la Géométrie analytique supérieure. Par H. ANDOYER. Volume I. Paris, Gauthier-Villars, 1900. 8vo, pp. vi + 508. Price, 15 francs.

IT would be asserting too much to say that M. Andoyer's first volume fulfills all reasonable expectations. In this volume only binary and ternary forms are treated, the quaternary field being reserved for a second, which is announced as already in press. All who read the author's introduction (lithographed) a few years ago must have expected to find in the present work a treatise not only compendious but also elementary. Compendious it certainly is, covering a surprisingly wide range, but the student beginning in this subject and reading it alone will find it impenetrable It is lucid, it is concise, but it is extremely condensed. Therein lies, however, its great merit As a work of reference, or as a syllabus to accompany a lecture course, it will supersede anything hitherto published in the same field. Its field is geometry—invariantive geometry, algebra taking the second place. Accordingly one cannot yet dispense with Salmon, Faa di Bruno, and Elliott. Nor is it intended to precede the study of projective geometry of curves; rather it presupposes a large amount of geometrical knowledge, and aims to recast and systematize it. To quote from the preface: "Je me suis proposé, en l'écrivant, d'exposer d'une façon didactique la théorie des formes et son interprétation géométrique générale"

Binary forms are treated in ten chapters, occupying 145 pages. After two excellent but too compact preliminary chapters, linear and quadratic forms and systems of forms are fully discussed, together with formal treatment of resultants and discriminants. Cubics, quartics, and quintics with their full covariant systems are given, the last only in list without any detail. All systematic discussion of *complete* form systems is excluded, the reader being referred to Gordan and Hilbert. Finally forms in two sets of variables are taken up, otherwise correspondences, and the metric geometry on a line. The chapter most novel in this binary division is that on the doubly quadratic form, or the (2, 2) correspondence on a line. Of course the problem of derived correspondences (2, 2) and the conditions for the occurrence of closed systems are of principal importance here, and I believe that no algebraic treatment has been accessible in textbooks heretofore. Andoyer derives the explicit fundamental invariants and covariants, the recursive formula expressing the equation of the $(n+1)$th derivative correspondence in terms of the nth and $(n-1)$th, and so ultimately in rational covariants of the original form. The results here obtained are directly applied in the chapter on Poncelet's polygons, in the ternary division.

Under ternary forms occur the necessary formal generalities, short chapters on homography and on conics (*la série quadratique*); then that just now alluded to upon the system of two conics, a welcome addition to the literature of the subject. Invariant conditions for closed Poncelet polygons of 3 and of 4 sides are given in rational functions of the four fundamental invariants, and the apparatus is furnished for calculating similar conditions for n sides. The derivation of these results by the aid of the elementary invariants of a binary biquadratic form is a decided improvement upon older methods.

Two bilinear forms and the quadratic correspondence are handled thoroughly, with a view to the quartic curve. Cubic and trilinear forms have a fairly full treatment (53 pages). The quartic curve is considered as the envelope of a variable conic whose parametric point describes a fixed conic: thus the bitangents and quadruply tangent conics are examined by the method of Cayley and Salmon. The two concluding chapters, apparently valuable, are concerned with metric geometry, general and special, *i. e.*, with a proper quadric and a degenerate quadric respectively as the absolute.

The thoroughly systematic execution of the work perhaps excuses the absence of any index, and the geometrical purpose naturally excludes any account of the work of Hilbert or of Deruyts; but a work of this magnitude, not professedly a cyclopædia, ought certainly to offer some perspectives and some outlook. The promise of a general bibliography in Volume II is hardly a substitute for the minute and particular citations that a student needs.—Enough of fault-finding! It is refreshing to read a treatise that makes perfectly free use of invariants of every degree of intricacy, with an eye solely to their significance and no concern about their explicit formulation. Let us notice that this last product of the nineteenth century on plane curve theory stops short of the quintic.

HENRY S. WHITE.

NOTES.

The second (April) number of Volume II of the *Transactions* of the American Mathematical Society contains the following papers: "Canonical forms of quaternary abelian substitutions in an arbitrary Galois field," by L. E. Dickson; "Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence," by M. Bôcher; "An elementary proof of a theorem of Sturm," by M. Bôcher; "On the determination of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by algebraic curves," by H. F. Stecker; "On the existence of a minimum of the integral $\int_{x_0}^{x_1} F(x, y, y')dx$ when x_0 and x_1 are conjugate points, and the geodesics on an ellipsoid of revolution; a revision of a theorem of Kneser's," by W. F. Osgood; "On the geometry of planes in a parabolic space of four dimensions," by I. Stringham.

The April number (second series, volume 2, number 3) of the *Annals of Mathematics* contains the following papers: "Sufficient conditions in the calculus of variations," by W. F. Osgood; "Lagrange's equation in the calculus of variations and the extension of a theorem of Erdmann," by J. K. Whittemore; "On some points in the theory of the hypergeometric function, expressed as a double circuit integral," by R. M. Hathaway; "A theorem in continued fractions," by D. N. Lehmer; "Note on the dual of a focal property of the inscribed ellipse," by R. E. Allardice; "A simplified solution of the cubic," by Emory McClintock.

At the regular meeting of the London mathematical society held on March 14, 1901, the following papers were read: "An account of some algebraical identities of simple arithmetical application," by Professor E. B. Elliott; "Preliminary notice concerning the stability of motion," by Professor A. E. H. Love; "On the composition of group characteristics," by Professor W. Burnside; "On the use of Cauchy's principal values in the double limit problems of the integral calculus," by Mr. G H. Hardy.

The following papers were read at the meeting of the Edinburgh mathematical society held on March 8, 1901: "Some elementary theorems regarding surds," by Professor G. Chrystal; "Note on the application of complex integration to the equations of conduction of heat," by Mr. J. Dougall.

The first fasciculus of volume IV of the Encyclopaedia of the mathematical sciences (see *Bulletin*, volume 7, page 101) is to appear very shortly. It will contain "Vector analysis," by Dr. M. Abraham of Göttingen, and Professor A. E. H. Love's two articles on "Hydrodynamics."

The first fasciculus of volume 9 of the *Jahresberichte* of the German mathematical association has just appeared. It contains an announcement that volume 10 will include a report by Professor H. Burkhardt entitled: "Die Ausbildung der Methode der Reihenentwickelungen an physikalischen Problemen."

The Spanish journal *El Progreso matematico*, edited by D. Zoel G. de Galdeano, Saragossa, ceased publication with the number for last December.

The first two numbers of a new Italian monthly, *Le Matematiche pure ed applicate* have appeared. It is edited by Professor C. Alasia of Oristano with the coöperation of twenty Italian and foreign mathematicians, and is published by S. Lapi in Città di Castello. The periodical will publish notes and memoirs on elementary and advanced mathematics, but preference will be given to articles on applied mathematics. The first number, which has twenty-four pages, begins with an article on continued fractions by the late Professor C. Hermite. Contributions may be sent to the American member of the board of editors, Professor G. B. Halsted, Austin, Texas.

The announcement is made that Professor A. von Oettingen of Leipsic has undertaken to edit Volume IV. of Poggendorff's Biographical Dictionary, which is to cover the period 1884–1900. Authors of works and articles in mathematics, physics, astronomy, meteorology, chemistry, mineralogy, geology, and geography have been asked to promote this undertaking by sending to Professor v. Oettingen biographical notices, and also information regarding books and articles published in periodicals which are not very widely known. The new volume will appear in 1902 or 1903, and will be published by J. A. Barth, of Leipsic.

Cambridge University. The Smith's prizes have been adjudged to G. H. Hardy, fourth wrangler in 1898, for his essay on "Definite integrals of discontinuous functions," and to J. H. JEANS, bracketed second wrangler in 1898, for his essay on "The distribution of molecular energy." P. V. BEVAN, fourth wrangler in 1899, received an honorable mention for his essay on "The influence of metallic media on light vibrations."

THE UNIVERSITY OF CHICAGO.—The following advanced mathematical courses, four hours weekly, are to be offered during the four quarters (*su*, *a*, *w*, *sp*) of the year beginning June 19, 1901:—By Professor E H. MOORE: Theory of functions of a real variable, with seminar (*a*, *w*, *sp*).—By Professor O. BOLZA: Theory of functions of a complex variable (*su*); Calculus of variations (*su*); Advanced algebra (*a*, *w*); Abelian functions, with seminar (*a*, *w*).—By Associate Professor H. MASCHKE: Theory of functions (*a*, *w*); Theory of the potential (*sp*); Theory of invariants (*a*); Twisted curves and surfaces, with seminar (*w*, *sp*).—By Assistant Professor H. E SLAUGHT: Differential equations (*su*); Advanced integral calculus (*a*, *w*, *sp*).—By Assistant Professor J. W. A. YOUNG: Pedagogy of mathematics (*su*); Solid analytics (*sp*).—By Assistant Professor L. E. DICKSON: Pure and analytic projective geometry (*su*); Theory of numbers, with introduction to congruence groups (*sp*).—By Dr. J. A. BOYD: Solid analytics and determinants (*su*).—By Dr. McDONALD: Hyperelliptic functions (*su*).

COLUMBIA UNIVERSITY.—The following advanced courses are offered during the academic year 1901–1902 by the departments of mathematics and mechanics, each course extending throughout the year:—By Professor F. N. COLE: Theory of groups, three hours.—By Professor T. S. FISKE: Advanced calculus, three hours; Functions defined by linear differential equations, three hours.—By Professor J. MACLAY: Theory of functions of a complex variable, three hours.—By Professor M. I. PUPIN: Electromagnetic theory of light, two hours; Bessel's functions and spherical harmonics, one hour.—By Professor R. S. WOODWARD: Advanced theoretical mechanics, two hours; Theory of the potential function, two hours.—By Dr. E. KASNER: Theoretical mechanics, three hours.—By Mr. C. J. KEYSER: Modern theories of geometry, three hours.—By Mr. H. B. MITCHELL: Differential equations, three hours. —By Mr. J. C. PFISTER: Theoretical mechanics, two hours.

The mathematical colloquium is held fortnightly.

CORNELL UNIVERSITY.—Among the courses announced for the summer session, July 5th to August 16th, are elementary courses in algebra, geometry, trigonometry, analytic geometry, and infinitesimal calculus, given by Professors TANNER, JONES, WAIT, and Dr. MURRAY, and the following advanced courses:—By Professor L. A. WAIT: Analytic geometry, based on Salmon's conic sections, three hours; Differential calculus, based on the treatises of Todhunter and Williamson, two hours.—By Dr. V. SNYDER: Projective geometry, an elementary course in Cremona's Projective geometry, five hours.—By Dr. D. A. MURRAY: Differential equations, five hours; Descriptive astronomy, five hours.—By Dr. G. A. MILLER: Integral calculus, based on the treatises of Todhunter and Williamson, five hours; Introduction to the theory of groups and the theory of numbers, five hours; History of mathematics, five hours.—By Dr. J. V. WESTFALL: Theory of functions of a complex variable, the elements of the theories of Cauchy, Riemann, and Weierstrass, three hours.

HARVARD UNIVERSITY.—The following advanced mathematical courses are offered during the academic year 1901–1902:—By Professor J. M. PIERCE: Quaternions (first course); Triangular coördinates and algebraic plane curves, especially cubics; † Application of quaternions to the theory of curves and surfaces; † Selected topics in quaternions.—By Professors W. E. BYERLY and B. O. PEIRCE: Trigonometric series, spherical harmonics, and potential function.—By Professor W. E. BYERLY: Calculus (second course); Dynamics of a rigid body.—By Professor W. F. OSGOOD: † Infinite series and products; Theory of functions (first course).—By Professor M. BÔCHER: † Algebra, properties of polynomials, invariants; † Introduction to partial differential equations; Functions defined by linear differential equations.—By Dr. C. L. BOUTON: † Theory of numbers; † Elementary differential equations.—By Mr. J. K. WHITTEMORE: Modern geometry; Hydrostatics, hydrokinetics.—By Mr. J. L. COOLIDGE: † Theory of equations, invariants; Non-euclidean geometry.

These courses will involve three lectures a week throughout the year, except those preceded by †, which involve about half this number of lectures. Professors BYERLY and OSGOOD, Dr. BOUTON, and Mr. COOLIDGE also offer courses in reading and research on Picard's Traité d'analyse, Calculus of variations, Theory of continuous groups, and Projective geometry, respectively.

The mathematical conference will meet twice a month.

YALE UNIVERSITY.—The following courses in mathematics are announced for the year 1901–1902 :—By Professor J. E. Clark : † Determinants, with applications to geometry and the theory of elimination, two hours ; † Differential equations, two hours.—By Professor J. W. Gibbs ; † Vector analysis, three hours ; † Advanced vector analysis, three hours ; Electricity and magnetism, one hour ; Thermodynamics and properties of matter, two hours.—By Professor W. Beebe : Celestial mechanics, three hours.—By Professor J. Pierpont : † Higher algebra, three hours ; Differential equations and function theory, three hours; Theory of functions, three hours.—By Professor P. F. Smith : Advanced differential geometry, two hours ; † Foundations of geometry, two hours.—By Professor H. A. Bumstead : Problems in mathematical physics, two hours.—By Dr. M. B. Porter : Advanced calculus, three hours ; † Selected topics in differential equations, three hours.—By Dr. W. A. Granville : † Differential geometry, three hours.—By Mr. E. B. Wilson : † Analytical mechanics, three hours ; † Projective geometry, three hours.

Courses preceded by † extend through only one half of the year.

The several German universities below offer during the summer semester, 1901, courses in mathematics as follows :

University of Berlin.—By Professor L. Fuchs : Applications of the theory of elliptic functions, three hours ; Theory of linear differential equations, four hours.—By Professor H. A. Schwarz : Integral calculus, four hours ; Theory of analytic functions, I, four hours ; Gauss's hypergeometric series, two hours ; Mathematical colloquium, two hours.—By Professor G. Frobenius : Theory of algebraic equations, II, four hours.—By Professor J. Knoblauch : Theory of space curves, one hour ; Theory of curved surfaces, four hours ; Elliptic functions, four hours. By Professor K. Hensel : Analytic geometry of plane and space, four hours ; Theory of probabilities, two hours ; Geometry of numbers and its application ; Colloquium, two hours.—By Professor R. Helmert : Figure of the earth, one hour ; Map projection, one hour.—By Professor R. Lehmann-Filhés : Differential Calculus, four hours, with exercises, one hour.—By Professor G. Hettner : Fourier's series and integrals, two hours.

University of Bonn.—By Professor L. Heffter : Infinitesimal calculus, four hours, with exercises, one hour ; Descriptive geometry, two hours, with exercises, three hours.—

By Professor R. LIPSCHITZ: Theory of numbers, four hours; Seminar, two hours.—By Professor H. KORTUM; Theory of algebraic equations, four hours; Seminar, two hours.

UNIVERSITY OF HALLE.—By Professor G. CANTOR: Theory of numbers, four hours; Definite integrals and differential equations, two hours; Seminar, two hours.—By Professor A. WANGERIN: Theory of potential and spherical harmonics, five hours; Seminar, two hours.—By Professor V. EBERHARD: Theory of Determinants, one hour: Plane analytic geometry, three hours.—By Dr. E. NEUMANN: Infinitesimal calculus, I, four hours, with exercises, one hour.

UNIVERSITY OF INNSBRUCK.—By Professor O. STOLZ: Theory of double integrals, two hours; Calculus of variations, one hour; Seminar on the foregoing subjects, one hour; Theory of functions of a complex variable according to Cauchy and Weierstrass (continued), with seminar, three hours.—By Professor W. WIRTINGER: Linear partial differential equations (continued), three hours; Euler's and the hypergeometric integrals, two hours; Seminar, two hours.—By Dr. K. ZINDLER: Descriptive geometry, four hours; Applications of calculus to geometry (continued), two hours.

UNIVERSITY OF VIENNA.—By Professor J. v. ESCHERICH: Calculus of variations, five hours; Mathematical statistics, three hours; Proseminar, one hour; Seminar, two hours.—By Professor L. GEGENBAUER: Theory of quadratic forms, two hours; Theory of numbers, II, three hours; Theory of probabilities, three hours; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: Elements of infinitesimal calculus for students of all faculties, II, five hours, with exercises, two hours; Proseminar, one hour; Seminar, two hours.—By Professor G. KOHN: Synthetic geometry (continued), four hours, with exercises, one hour.—By Dr. V. SERSAWY: Mathematics of insurance, two courses of three and four hours respectively.—By Dr. A. TAUBER: Analytic geometry of space, three hours; Mathematics of insurance, three hours, with exercises, two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics.—By Dr. R. D. v. STERNECK: On the number of prime numbers between given limits, one hour.—By Dr. J. A. GMEINER: Quaternions. with applications to the motion of points, one hour.

A COURSE of six lectures on "British mathematicians of the nineteenth century" was delivered by Dr. ALEXANDER

MACFARLANE at Lehigh University, April 12–23. The life and work of the following mathematicians was presented: George Peacock (1791–1858), Augustus De Morgan (1806–1871), W. R. Hamilton (1805–1865), George Boole (1815–1864), Arthur Cayley (1821–1895), W. K. Clifford (1845–1879).

DR. J. C. FIELDS has recently given at the University of Toronto a course of lectures on "The quadrature of the circle."

PROFESSOR J. W. GIBBS, of Yale University, has been elected an honorary member of the London physical society.

PROFESSOR G. HUMBERT of the Ecole polytechnique has been elected a member of the section of geometry of the Paris academy of sciences to fill the vacancy caused by the death of HERMITE.

PROFESSOR ASAPH HALL, U. S. Navy, who for the last five years has been lecturer on celestial mechanics at Harvard University, has resigned his position there, and will spend the next year or two in European travel.

IT is reported that the faculty of the University of Munich has nominated Professors D. HILBERT, A. BRILL, and A. VOSS to fill the vacancy caused by the resignation of Professer G. BAUER.

DR. E. A. ENGLER, professor of mathematics in Washington University, St. Louis, Mo., has been elected president of the Worcester Polytechnic Institute, to succeed Dr. T. C. MENDENHALL, who retires on account of ill-health.

AT Columbia University, Dr. JAMES MACLAY has been promoted to an associate professorship of mathematics, and Mr. J. C. PFISTER to an instructorship in mechanics. Dr. G. H. LING, of Wesleyan University, and Mr. WILLIAM FINDLAY, of the University of Chicago, have been appointed tutors in mathematics in Columbia University.

MR. H. W. KUHN, of Cornell University, has been appointed assistant professor of mathematics in Ohio State University.

DR. R. E. MORITZ, of the University of Nebraska, has received leave of absence and will spend a year in study in Germany.

PROFESSOR H. A. ROWLAND, Johns Hopkins University, died at Baltimore, April 16.

THE death is announced of DR. J. T. DUFFIELD, professor of mathematics in Princeton University since 1847.

THE deaths are also announced of M. T. MOUTARD, an eminent French mathematician and engineer and honorary examiner at the Ecole polytechnique; of Dr. P. M. POKROWSKI, professor of mathematics at Kiew; and of Dr. F. MELDE, professor of physics and director of the mathematical and physical institute of the University of Marburg.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

COLLINS (J. V.). An elementary exposition of Grassmann's Ausdehnungslehre, or theory of extension. Springfield, Mo., Dixon, 1901, 8vo. 2+46 pp. (Reprinted from the *American Mathematical Monthly*, Vols 6 and 7.)

KIEPERT (L.). Grundriss der Differential- und Integralrechnung. Teil I: Differentialrechnung. 9te Auflage des gleichnamigen Leitfadens von M. Stegemann. Hannover, Helwing, 1901. 8vo. 17 + 750 pp. Cloth. M. 12.00

MANNING (H. P.). Non-euclidean geometry. Boston, Ginn, 1901. 12mo. 4 + 95 pp. Cloth. $0.75

MOLK (J.). See TANNERY (J.).

NICHOLS (E. W.). Differential and integral calculus, with applications; for colleges, universities and technical schools. Boston, Heath, 1900. 12mo. 405 pp. Half-leather. $2.00

STEGEMANN (M.). See Kiepert (L.).

TANNERY (J.) et MOLK (J.). Eléments de la théorie des fonctions elliptiques. (En 4 volumes.) Vol. 4: Calcul intégral, partie 2e, et applications. Fascicule I. Paris, Gauthier-Villars, 1900. 8vo. Pp. 1—164. Vol. 4 complete Fr. 9.00

WIENER (H.). Die Einteilung der ebenen Kurven und Kegel dritter Ordnung in 13 Gattungen. Teil I: Die Konfiguration der neun Wendepunkte einer ebenen Kurve dritter Ordnung. Teil II: Der syzygetische Buschel von Kurven dritter Ordnung. Halle, Schilling, 1901. 8vo. 6 + 34 pp. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von M Schilling in Halle a. S.; neue Folge, No. 2.) M. 1.60

II. ELEMENTARY MATHEMATICS.

ALASIA (C.). La recente geometria del triangolo. Città di Castello, Lapi, 1900. 8vo. 16 + 339 pp. Fr. 3.00

Alasia (C.). Relazioni fra gli elementi di un triangolo piano; 566 formule raccolte ed ordinate. Città di Castello, Lapi, 1900. 8vo. 35 pp.

Bergmann (F.). See Rossmanith.

Boyden (W. C.). Primeros pasos en álgebra; traducido y adaptado al idioma castellano. New York, Silver, Burdett, & Co. [1900]. 12mo. 194 pp. Cloth. $0.60

Buck (R. C.). Manual of trigonometry; for use, more especially, of young sailors and officers in the merchant navy. Revised edition. London, Griffin, 1901. 12mo. 124 pp. Cloth. (Nautical series.) 3s. 6d

Dekker (P.). See Niemöller (F.).

Euclid. See Simon (M.).

French (C. H.) and Osborn (G.). First year's algebra; with or without answers. London, Churchill, 1901. 12mo. Cloth. 1s. 6d.

Gabrielli (A.). Nuovo metodo per la risoluzione delle equazioni ad una incognita del secondo, del terzo grado e di grado superiore. Milano, Pirola, 1901. 8vo. 26 pp. Fr. 1.50

Jelinek (L.). Logarithmische Tafeln für Gymnasien und Realschulen. 4te Auflage. Samt Anleitung. Wien, Pichler, 1900. 8vo. 4 + 157 pp. Cloth. M. 1.50

J. (F.). Compléments de trigonométrie et méthodes pour la résolution des problèmes. Paris, Poussielgue [1901]. 18mo. 12 + 843 pp. (Cours de mathématiques élémentaires.)

Kühl (I. H.). Grundriss der Geometrie; ein Leitfaden fur den Unterricht. Teil I. Planimetrie. 2te Auflage. Dresden, Kuhtmann, 1900. 8vo. 4 + 95 pp. M. 1.40

Matriculation mathematics; model answers. January, 1895, to January, 1901. London, Clive, 1901. 12mo. 156 pp. (University Tutorial Series.) 2s.

Métral (A.). La trisezione geometrica dell'angolo, con traduzione francese della parte principale. Roma, Pistolesi, 1900. 8vo. 32 pp., 1 plate.

Milne (W. J.). Academic algebra. New York, American Book Company [1901]. 12mo. 2 + 444 pp. Half-leather. $1.25

Niemöller (F.) und Dekker (P.). Arithmetisches und algebraisches Unterrichtsbuch. Für den mathematischen Unterricht in der Mittel- und Oberstufe hoherer Lehranstalten nach den Bestimmungen der preussischen Lehrpläne von 1892 bearbeitet. (In 4 Heften.) Heft 3: Pensum der Obersekunda. Breslau, Hirt, 1901. 8vo. 80 pp. Boards. M 1.25

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LI R RY

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 27, 1901. Thirty-four persons were present at the two sessions, including the following thirty-one members of the Society:

Mr. Joseph Allen, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Mr. F. A. Giffen, Dr. W. A. Granville, Miss Ida Griffiths, Professor James Harkness, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. G. H. Ling, Dr. Emory McClintock, Dr. James Maclay, Dr. G. A. Miller, Professor Frank Morley, Professor W. F. Osgood, Miss Ida Schottenfels, Professor C. A. Scott, Dr. Virgil Snyder, Dr. H. F. Stecker, Professor J. H. Tanner, Professor H. D. Thompson, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Professor R. S. Woodward.

Vice-President Thomas S. Fiske occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. C. W. McG. Black, Harvard University, Cambridge, Mass.; Dr. S. E. Slocum, University of Cincinnati, Cincinnati, Ohio. Two applications for membership were received.

To relieve the increasing burden of administration, the office of Assistant Secretary was created, and filled by the appointment of Dr. Edward Kasner, to serve until February, 1902.

The library of the Society, which at present consists mainly of some five hundred unbound volumes of journals received as exchanges, is about to be deposited in the library of Columbia University, under an agreement by which the University undertakes to bind, catalogue, and care for the books now on hand and all future additions, and to make them easily accessible to the members of the Society. Arrangements will be made by which the books may be temporarily loaned to members living at a distance. The library is to be kept as a separate collection, duplicating as far as may be the general University library, and aiming to become as complete as possible in itself. The title to the books remains in the Society, which reserves the right to withdraw them under agreed conditions.

The following papers were read at this meeting:

(1) Dr. W. A. Granville: "Invariants of some m-gons under certain projective Lie groups in the plane."

(2) Dr. Edward Kasner: "The algebraic potential surfaces."

(3) Professor F. Morley: "On the real foci of algebraic curves."

(4) Mr. George Peirce: "A curious approximate construction for π."

(5) Professor E. W. Hyde: "On a surface of the sixth order, which is touched by all screws belonging to a three-conditioned system."

(6) Professor L. E. Dickson: "The hyper-orthogonal groups."

(7) Professor E. W. Brown: "On least action and minimal surfaces."

(8) Professor W. H. Metzler: "On certain aggregates of determinant minors."

(9) Professor E. B. Van Vleck: "On the convergence of continued fractions with complex elements; supplementary note."

(10) Professor W. F. Osgood: "On a fundamental property of a minimum in the calculus of variations and the proof of a theorem of Weierstrass."

(11) Professor E. O. Lovett: "The geometry of quadrics."

(12) Professor E. O. Lovett: "The differential geometry of n-dimensional space."

(13) Dr. G. A. Miller: "On the groups generated by two operators."

(14) Dr. Edward Kasner: "The relations between the angles of any number of lines in n-space."

(15) Dr. L. P. Eisenhart: "Isothermal conjugate systems of lines on surfaces."

(16) Dr. E. J. Wilczynski: "Geometry of a simultaneous system of two linear homogeneous differential equations of the second order."

(17) Dr. H. F. Blichfeldt: "A new determination of the primitive continuous groups in two variables."

Mr. Peirce's paper was communicated to the Society by Professor E. W. Brown. In the absence of the authors, the papers of Mr. Peirce, Professor Hyde, Professor Dickson, Professor Brown, Professor Metzler, Professor Lovett, Dr. Eisenhart, Dr. Wilczynski, and Dr. Blichfeldt were read by title. The papers of Mr. Peirce and Dr. Miller will appear in the Bulletin. Abstracts of the other papers are given below.

Dr. Granville finds the invariant functions of certain configurations under the substitutions of some of the important subgroups of the general projective group in the plane. Problems of this sort are among the most important and interesting in Lie's theory of continuous groups. First an outline is given of the theory followed in solving the problems. Then the invariant functions of a quadrangle under the substitutions of the group

$$\boxed{xp,\ yp,\ xq,\ yq,\ x^2p + xyq,\ xyp + y^2q}$$

are found. The results do not agree with those found by Professor Lovett in the *Annals of Mathematics*, volume 12, where the same problem is proposed. The remaining four problems solved are of the same nature.

The surfaces considered by Dr. Kasner in his first paper are obtained by equating the rational integral potential functions $\varphi(x, y, z)$ to zero; their analytic property is expressed by the equation

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0.$$

It is shown that the surfaces may be defined geometrically by their relation to the imaginary circle at infinity: the surfaces are apolar to this circle when the latter is considered as a degenerate surface of the second class. The potential surfaces of the second order are rectangular hyperboloids, *i. e.*, their asymptotic cones contain triples of mutually orthogonal generators. The polar quadrics of any potential surface are rectangular hyperboloids; conversely, a surface is potential when all its polar quadrics are rectangular hyperboloids. Potential cylinders and cones are considered in detail. The results are finally extended to n-dimensional space.

The point of Professor Morley's note was that, given any line equation of an algebraic curve

$$f(\xi_1, \cdots, \xi_n) = 0,$$

where ξ_r is the distance from a fixed point a_r to the moving line, the real foci are given by writing in the terms of highest order $x - a_r$ for ξ_r, where $x - a_r$ is the stroke from a fixed point to a focus. Some applications of this simple rule were given.

Any three screws determine a system of screws in space consisting of the "totality of the generators of one system of a skew conicoid which changes in accordance with the variation of a parameter involved in its equation." Professor Hyde assumes a particular arrangement of the determining screws, viz.; they are mutually perpendicular, and have a common point. The envelope of the before-mentioned skew conicoid is found, which will be touched by all the screws of the system. It is a surface of the sixth order whose constants are functions of the differences of the pitches of the determining screws, and is shown to be reducible to a comparatively simple form in which only two of these differences appear. The peculiarities of the surface are discussed, and three diagrams given, illustrating the different forms which it may assume.

In a paper published in the *Mathematische Annalen*, volume 52, pp. 561–581, Professor Dickson showed that the structures of the groups defined by invariants

$$\sum_{i=1}^{m} \lambda_i \xi_i^r \qquad (r > 2)$$

in any Galois field depend upon the structure of the group with the invariant

$$\sum_{i=1}^{m} \xi_i^{p^s+1}$$

in the $GF[p^{2s}]$. The latter group is called the hyperorthogonal group H, since it includes the orthogonal group as a subgroup. It may be represented as a transitive substitution group on N_c symbols

$$\lambda_1 \xi_1 + \cdots + \lambda_m \xi_m$$

where $\lambda_1, \cdots, \lambda_m$ are any marks, not all zero, satisfying

$$\lambda_1^{p^s+1} + \cdots + \lambda_m^{p^s+1} = c = \text{constant.}$$

For any $c \neq 0$, we have

$$N_c = p^{s(2m-1)} - (-1)^m p^{s(m-1)};$$

while, for $c = 0$, N_c has the value

$$\omega_m \equiv [p^{ms} - (-1)^m]\,[p^{(m-1)s} - (-1)^{m-1}].$$

The substitutions of H which multiply every index by the same constant form an invariant subgroup of H. The quotient-group $HO(m, p^{2s})$ is a simple group except when $m = 2$, $p^s = 2$; $m = 2$, $p^s = 3$; $m = 3$, $p^s = 2$. Except when $m = 2$, $p^s = 2$, the quotient group may be represented as a transitive substitution group on $N_c \div (p^s + 1)$ letters for $c \neq 0$, as well as upon $\omega_m \div (p^{2s} - 1)$ letters for $c = 0$. The latter number is the smaller except for $p^s = 2$, m even, $m > 2$, when the former number is $\frac{1}{3}(2^{2m-1} - 2^{m-1})$. For $m = 2$, the smaller number is $p^s + 1$, a result in accord with the known isomorphism of $HO(2, p^{2s})$ with the linear fractional group in the $GF[p^s]$.

The paper makes a detailed study of the characteristic equations of hyperorthogonal substitutions. If x be one root, then is x^{-p^s} also a root. Inversely, given an equation having this property, we can determine a hyperorthogonal substitution having it as its characteristic equation. Such an equation is never irreducible in the $GF[p^{2s}]$ if $m \neq 3$; it never decompose into linear factors and a single irreducible factor of degree $\neq 3$. If $f_t(x)$ be one irreducible factor, there exists a complementary irreducible factor

$$x^t\{f_t(x^{-p^{-s}})\}^{p^s}$$

of equal degree t. Thus, for $m = 4$, if $x^2 - ax + \beta$ be an irreducible factor, then is also

$$x^2 - xa^{p^s}\beta^{-1} + \beta^{-1},$$

where β necessarily belongs to the included $GF[p^s]$. The greater part of the paper is devoted to the reduction of hyperorthogonal substitutions to canonical forms within the group and to their distribution into complete sets of conjugate substitutions. The results are exhaustive for the case $m = 3$. For example, when $p^s = 3$, $m = 3$, the group H is a simple group of order 6048, containing exactly 14 non-conjugate types of substitutions whose periods are 1, 2, 3, 4, 6, 7, 8, 12.

Professor Brown showed that the problem of the motion of a particle in a plane is reducible to the discovery of a certain cylindrical surface whose area between two generating lines is a minimum. A similar result is also shown for the case of a particle whose motion is referred to axes moving in the plane with uniform velocity.

In 1888 (*Proceedings of the Royal Society of Edinburgh*, pp. 99–105) Dr. T. Muir showed that a linear relation exists between certain minors of a centro-symmetric determinant similar to Kronecker's relation between the minors of an axi-symmetric determinant; and in 1900 (same journal, pp. 142–154) he gave two theorems connecting the minors of any determinant, the first of which reduces to Kronecker's relation on imposing the conditions for axi-symmetry, and the second of which reduces to his 1888 relation on imposing the conditions for centro-symmetry. Professor Metzler extends these two theorems given by Dr. Muir so as to include his as special cases and gives a whole series of types of linear relations between the minors of a centro-symmetric determinant, Muir's being one type of the series. The paper also gives the number of relations of each type.

Professor Van Vleck presented a note supplementary to his paper read at the February meeting. The note will be embodied in the paper,, which will appear in the *Transactions.*

Professor Osgood's paper, which will appear in the *Transactions*, is in abstract as follows: The integral

$$I=\int_{x_0}^{x_1} F(x, y, y')dx$$

or, in the parametric form,

$$I=\int_{t_0}^{t_1} F(x, y, x', y')dt$$

is said to be made a minimum by the functions $x=\varphi(t)$, $y=\psi(t)$ corresponding to the curve C, if its value J corresponding to this curve is less than its value I corresponding to any other curve $\overline{C}$ lying in a neighborhood T of C and satisfying the boundary conditions of the problem. Weierstrass has given a sufficient condition for a minimum. The question, however, presents itself: Assuming that Weierstrass's sufficient condition is fulfilled, so that no curve $\overline{C}$, distinct from C, of the class of curves admitted to consideration will give the integral I so small a value as J, may there not still be a set of curves belonging to this class, $\overline{C}_1, \overline{C}_2, \cdots$ which do not cluster about C as their limit, and which have the property that, if $I_1, I_2, \cdots$ denote respectively the values of the integral I formed for these curves,

$$\lim_{n=\infty} I_n = J\,?$$

This question is answered in the negative. It is shown that, if $\mathfrak{T}$ denotes an arbitrarily small neighborhood of C lying wholly within T, the lower limit of I formed for all the curves $\overline{C}$ of the class in question that do not lie wholly within $\mathfrak{T}$ is greater than J by a positive quantity ε, varying for different regions $\mathfrak{T}$, but constant for any given region; so that

$$I \geqq J + \varepsilon, \qquad \text{when } \overline{C} \text{ does not lie wholly in } \mathfrak{T}.$$

By means of this result it is possible to extend the class of curves $\overline{C}$ with reference to which the integral I possesses the minimum property and thus establish the existence of a more general minimum. Making the generalized definition of the length of a curve his point of departure, Weierstrass inscribed in an arbitrary curve $\overline{C}$ that satisfies the boundary conditions, is continuous, and lies within T, a polygon and formed the sum

$$S = \sum_{\iota=0}^{n-1} F\left(x_\iota, y_\iota, \frac{\Delta x_\iota}{\Delta t_\iota}, \frac{\Delta y_\iota}{\Delta t_\iota}\right) \Delta t_\iota = \sum_{\iota=0}^{n-1} F(x_\iota, y_\iota, \Delta x_\iota, \Delta y_\iota).$$

If S converges toward one and the same limit, no matter how the vertices of the polygon be chosen, provided merely that all its sides converge toward 0, this limit is taken as the generalized limit $\mathfrak{J}$ and the class of curves admitted to consideration is enlarged so as to include $\overline{C}$. Weierstrass states the theorem that $\mathfrak{J} > J$ if $\overline{C}$ is distinct from C. This theorem is proven in the present paper and it is furthermore shown that

$$\mathfrak{J} \geqq J + \varepsilon \qquad \text{when } \overline{C} \text{ does not lie wholly in } \mathfrak{T}.$$

The theory of the group of euclidean motions yields the notions of ordinary n-dimensional geometry. In his first paper Professor Lovett shows that it can also be made to yield the elements of this geometry. The group is extended relative to the coefficients of the equation of the general quadric. An application of Euler's theorem facilitates the construction of the invariants. In particular it appears that the squares of the semi-axes of the quadric

$$\sum_{1}^{n}\sum a_{ij} x_i x_j + 2\sum_{j=1}^{n} a_{j,\,n+1} x_j + a_{n+1,\,n+1} = 0$$

are given by the equation

$$I_{n+1}^{n} + \sum_{1}^{n-1} I_n^{\iota} I_\iota I_{n+1} x^{\iota} + I_n x^n = 0,$$

where

$$I_k = \sum_1^n {}_{\iota_1\iota_2\cdots\iota_k} |a_{\iota_1\iota_1}, a_{\iota_2\iota_2}, \cdots, a_{\iota_k\iota_k}|.$$

Every space of k dimensions within the n-dimensional space has its own differential geometry. To determine its elements, analogous to those of Gauss for the differential geometry of surfaces in ordinary space, it is only necessary to construct the nth extension of the group of euclidean motions relative to k independent parameters which are assumed to be invariant under the transformations of the group.

Professor Lovett's second paper effects these extensions for the values 1 and $n-1$ of k, and constructs the corresponding differential invariants. In particular, it appears that the following relations are the generalizations for four dimensional space of the three well-known relations of Gauss, Mainardi, and Codazzi among the Gaussian fundamental quantities of the first and second order:

$$2 \cdot 12_{12} - 11_{22} - 22_{11} = \varphi_1, \quad 2 \cdot 23_{23} - 22_{33} - 33_{22} = \varphi_2,$$

$$2 \cdot 31_{31} - 33_{11} - 11_{33} = \varphi_3,$$

$$12_{33} - 13_{23} = \varphi_4, \quad 13_{22} - 12_{23} = \varphi_5, \quad 13_{12} - 12_{13} = \varphi_6,$$

$$12_2 - 22_1 = \psi_1, \quad 23_3 - 33_2 = \psi_2, \quad 22_3 - 23_2 = \psi_3,$$

$$11_2 - 12_1 = \psi_4, \quad 12_3 - 23_1 = \psi_5, \quad 11_3 - 31_1 = \psi_6,$$

$$12_3 - 31_2 = \psi_7, \quad 33_1 - 31_3 = \psi_8,$$

where

$$\ddot{y}_{kl} = E_{ij_{u_k u_l}}, \qquad \ddot{y}_k = F_{ij_{u_k}},$$

$$E_{ij} = \sum_1^n {}_k \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j}, \qquad F_{ij} = \sum_1^n {}_k X_k \frac{\partial^2 x_k}{\partial u_i \partial u_j},$$

$$X_i |E_{11}, E_{22}, \cdots, E_{n-1\,n-1}|^{\frac{1}{2}} = M_i,$$

in which M_i is the determinant formed by suppressing the ith column of the matrix

$$\| x_{1_{u_1}}, x_{2_{u_2}}, \cdots, x_{n_{u_n}} \|,$$

and the φ's and ψ's are functions of the first and second order differential invariants whose forms are easily constructed. The theory of this group gives the whole theory of k-parametric displacements in a space of n-dimensions, as a future note will show.

Dr. Kasner's second paper shows that, in space of n dimensions, the angles of any number of lines less than $n+1$ are all independent; while the $\frac{1}{2}n(n+1)$ angles of $n+1$ lines $L_1, L_2, \cdots, L_{n+1}$ are connected by the single relation

$$\begin{vmatrix} 1 & (1,2) & \ldots\ldots & (1, n+1) \\ (2,1) & 1 & \ldots\ldots & (2, n+1) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (n+1, 1) & (n+1, 2) & \ldots\ldots & 1 \end{vmatrix} = 0,$$

where (i, j) denotes the cosine of the angle L_i, L_j. The relations for any number of lines are of the same type. For ordinary space this gives the known relation between the sides and diagonals of a spherical quadrilateral, and the relation between the dihedral angles of a tetrahedron.

Dr. Eisenhart's paper is, in abstract, as follows: An extension of the term isothermal-conjugate system of lines, as given by Bianchi to those systems on a surface of positive curvature for which as parametric curves the second fundamental form reduces to $\lambda(du^2 + dv^2)$, is made to those conjugate systems on surfaces of negative curvature for which this form reduces to $\lambda(du^2 - dv^2)$, where λ is a function of u and v. Results are found for these systems very similar to those given by Bianchi for the former case. A general study is made of these lines on both kinds of surfaces; the determination of these systems on surfaces of positive curvature is effected by methods similar to those used by Bianchi in the study of isothermal orthogonal systems. A geometrical interpretation of isothermal-conjugate systems leads to the theorem that the surfaces obtained by reciprocal radii vectores from surfaces whose lines of curvature are isothermal-conjugate are of the same kind. The greater part of the paper is devoted to a study of surfaces whose lines of curvature are isothermal-conjugate, and it is shown that the general problem of determining such surfaces depends upon the integration of a differential equation of the fourth order, very similar to the equation found by Darboux in his study of isothermic surfaces. It is noted that the surfaces whose lines of curvature are at the same time isothermal and isothermal-conjugate have an isothermal spherical representation, and from this is deduced the theorem that the sphere and minimal surfaces are the only surfaces of constant mean curvature whose lines of curvature are isothermal-conjugate; also that the quadric surfaces have isothermal-conjugate lines of curvature.

It is shown that the plane, sphere, cyclides of Dupin, and surfaces of revolution are the only surfaces which together with their parallels have an isothermal conjugate system of lines of curvature; and that the surfaces of constant total curvature, the surfaces of revolution, and the surfaces whose radii satisfy two definite equations of condition (not given here) are the only Weingarten surfaces possessing the above property. Associated with a surface with isothermal-conjugate systems of lines of curvature is a second surface with isothermal lines of curvature, corresponding to the former by parallelism of tangent planes; it can be found by quadratures. In particular, when the first surface is of constant total curvature, the second is of constant mean curvature and parallel to the former. The paper closes with a discussion of surfaces with plane lines of curvature in both systems which at the same time are isothermal-conjugate. Solutions of this problem are furnished by surfaces of revolution, certain moulded surfaces, cyclides of Dupin, minimal surfaces of Bonnet and Enneper, and other surfaces whose equations are given.

In a paper recently published in the *Transactions*, Dr. Wilczynski has initiated a theory of invariants of a system of linear differential equations. In the present paper, the author considers a remarkable relation between this theory and line geometry. If (y_i) and (z_i) $(i = 1, 2, 3, 4)$ form a fundamental system of simultaneous solutions of the system of differential equations

$$(1) \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

and y_i and z_i be interpreted as the homogeneous coördinates of two points in space, it is seen that (1) defines two curves C_y and C_z as its integral curves. The points (y_i) and (z_i) of these curves, which correspond to the same value of the independent variable x, are joined by a straight line L_{yz}, and these straight lines generate a ruled surface S, called the integrating ruled surface of system (1), which, it is shown, can never be a developable surface.

The most general transformations which convert (1) into another system of the same kind are

$$(2) \qquad x = f(\xi), \quad y = \alpha\eta + \beta\zeta, \quad z = \gamma\eta + \delta\zeta,$$

where f, α, β, γ, δ are arbitrary functions of ξ. It is noted that these transformations leave invariant the ruled surface

S, merely converting the two integral curves into any other two curves on this surface. Invariant relations, it is shown, give projective properties of the integrating ruled surface. The differential equation of the sixth order, satisfied by the line coördinates of a generator of the ruled surface, is set up in a normal form. Then the necessary and sufficient conditions are discussed under which the generators of the ruled surface belong to a linear complex, either general or special, to a linear congruence with distinct or coincident directrices, and finally the conditions under which they form a surface of the second order. The reduction of the system to the semi-canonical form is found to be equivalent with the geometrical problem of finding the curved asymptotic lines on the surface, and a new proof of Serret's theorem about such lines is given.

The primitive continuous group of point transformations in two variables can, by a proper choice of the variables, be transformed into *projective groups* of the plane, a result Lie obtains after determining the canonical forms of the primitive groups. This fact can, however, be established from the general properties of such groups, and it is shown in Dr. Blichfeldt's paper, which will appear in the *Transactions*, that this point of view gives rise to a new determination of these primitive groups.

EDWARD KASNER,
Assistant Secretary.

COLUMBIA UNIVERSITY.

THE APRIL MEETING OF THE CHICAGO SECTION.

A REGULAR meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Saturday, April 6, 1901. The following members were present:

Professor W. W. Beman, Professor Oskar Bolza, Professor D. F. Campbell, Professor E. W. Davis, Professor Thomas F. Holgate, Professor E. H. Moore, Dr. F. R. Moulton, Miss Ida M. Schottenfels, Professor H. E. Slaught, Mr. Burke Smith.

The President of the Society, Professor E. H. Moore, occupied the chair. The following papers were presented and read:

(1) Dr. F. R. MOULTON: "On the validity of the method of computing absolute perturbations."

(2) Professor GEORGE C. CHATBURN: "A theorem in arrangements."

(3) Professor E. W. DAVIS: "A definition of mathematical probability."

(4) Dr. JACOB WESTLUND: "Note on multiply perfect numbers."

(5) Professor F. MERTENS: "Zur linearen Transformation der θ-Reihen."

(6) Dr. J. H. McDONALD: "The reduction of hyperelliptic integrals of genus two to elliptic integrals, by a transformation of order four."

(7) Professor L. E. DICKSON: "Representation of linear groups as transitive substitution groups."

(8) Dr. G. A. MILLER: "Determination of all the groups of order p^m which contain the abelian group of the type $(m - 2, 1)$, p being any prime number."

(9) Mr. ROBERT E. MORITZ: "A generalization of the differentiation process."

(10) Mr. ROBERT E. MORITZ: "On ratients, with an extension of the ordinary calculus."

Professor Chatburn's paper and the two papers by Mr. Moritz were presented to the Society through Professor Davis, and in the absence of the authors were read by him; Professor Mertens's paper was presented through Professor Moore; in the absence of Dr. Westlund his paper was read by the Secretary. Dr. McDonald was introduced to the Section by Professor Bolza. Abstracts of the papers follow.

It is generally understood, and indeed it is usually stated in works on celestial mechanics,* that the methods used by astronomers are only approximate, or else that the convergence of the series employed is assumed. The former means to a mathematician that the functions from which the perturbations are computed do not represent the actual perturbations with rigor; the latter leaves the validity of the whole process in doubt. In Dr. Moulton's paper it is asked whether the power series in the masses used in computing the perturbations are convergent, and whether they rigorously represent these perturbations. It is answered that, under certain conditions which are fulfilled in case of the major planets, the series in the masses, in actual use by astronomers, are absolutely convergent for all values of the time not greater in numerical value than a fixed limit which can be determined, and that these series represent the perturbations exactly. This limit within which the series are certainly convergent is doubtless much smaller than the true radius of convergence. The results are equally true in Laplace's method of computing the absolute perturbations of the coördinates, and in Lagrange's method of the variation of parameters.

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* Dziobek's Planeten-Bewegung, p. 167; Tisserand's Mécanique céleste, vol. 1, pp 190 and 193–4; Brown's Lunar theory, preface, p. vi.

The method of proof is an adaptation of Cauchy's calcul des limites,* which has been used with great success in the well known works of Fuchs, Frobenius, Weierstrass, Poincaré, and many others, in the theory of differential equations. The particular modification used here is the same extension as that employed by Poincaré in his memoir in the *Acta Mathematica,* volume XIII. Indeed, Poincaré's results are closely related to those attained here; but, having another object in view, his investigation does not regard the validity of the precise method used by astronomers.

The second part of the paper is devoted to a discussion of the geometrical and physical meaning of the terms of the various orders with respect to each mass separately. It is shown that there is a perfect correspondence between the terms of the power series and the successive corrections to the elliptic orbits which, from a general mathematical point of view, seem to lead most naturally toward the true orbit. The conclusions are that the methods used by astronomers are, under certain limitations, perfectly valid, and that "successive approximations" means successive approximations toward the true numerical values by means of series which represent the functions exactly.

Professor Chatburn's theorem in arrangements gives a method by which c things (say cards numbered consecutively) can be divided into groups, and then rearranged group by group, redivided and rearranged for n or more times, so that the final position of any one of the things shall be that given by the terms of an arithmetical progression in which unity is the first term, c the last term, and p the number of terms.

Professor Davis offered the following definition for mathematical probability: the direct probability that the occasion A shall lead to the event B is the value toward which tends the ratio of the number of A's that are followed by B, to the total number of A's, as these recur without limit. He also stated a corresponding definition for inverse probability.

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* *Comptes rendus,* vol. 14.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the multiplicity being the quotient, Dr. Westlund shows in his note that the only numbers of multiplicity 3 of the form $m = p_1^{a_1}\ p_2^{a_2}\cdot p_3$ where p_1, p_2, p_3 are three distinct primes, are the two numbers $2^3\cdot 3\ \ 5$ and $2^5\cdot 3\cdot 7$.

Professor Mertens makes a direct and elegant determination of the twenty-fourth root of unity, fundamental in the theory of the linear transformations of the θ-series. His paper will be published in the *Transactions*.

The reduction of hyperelliptic integrals of genus two to elliptic integrals is effected by a special involution of order four containing a form which is the square of a quadratic. From the general Weierstrass-Picard theorem it is known that if there exists one reducible integral, there exists also a second having the same irrationality and reducible by a transformation of the same order. The problem treated in Dr. McDonald's paper is: Given one integral, to find the other. It is first shown that the special involution of order four may be uniquely determined by a cubic and a quadratic form. Among the covariants of this system of cubic and quadratic there exists another cubic and quadratic having with the first mutual relations analogous to, but more perfect than, the relations between a single cubic form and its cubic covariant. The involution which is determined by this second cubic and quadratic is one which will deduce a second hyperelliptic integral having the same irrationality as the first but a different numerator, which must be the one indicated by the Weierstrass-Picard theorem. The relations between the two numerators of reducible integrals and the corresponding involutions are a generalization of those found by Professor Bolza for the reduction of order three, and which also exist for the reduction of order two.

In Professor Dickson's paper, which is to be published in the *American Journal of Mathematics*, it is pointed out that the group of all linear homogeneous substitutions S on m indices with coefficients in the $GF\ [p^n]$ may be represented as a transitive substitution group on $p^{nm} - 1$ letters; the simple group defined by taking the substitutions S fractionally may be represented as a doubly transitive substitution group on $(p^{nm}-1) / (p^n-1)$ letters. In the former case we may employ as the $p^{nm}-1$ letters the totality of functions $\lambda \equiv \lambda_1\xi_1 + \lambda_2\xi_2 + \cdots + \lambda_m\xi_m$, in which the λ_i run independently through the series of marks of the field, the case $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ being excluded.

For a subgroup defined by an invariant $\psi(\xi_1, \cdots, \xi_m)$, the above representation leads to an intransitive substitution group. To illustrate the method of obtaining a transitive representation, consider the orthogonal group. If an orthogonal substitution replaces ξ_1 by $\omega \equiv \alpha_1\xi_1 + \cdots + \alpha_m\zeta_m$ then $\alpha_1^2 + \cdots + \alpha_m^2 = 1$; inversely, for any such set of marks α_i, the orthogonal group contains a substitution replacing ξ_1 by ω. Under an orthogonal substitution the above functions λ are permuted in such a way that the totality of functions λ in which $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2$ is a constant mark c are permuted transitively. Except when $\psi S = 3$, the case $c = 0$ leads to a representation upon fewer letters than any case $c \neq 0$. For $c = 0$, we combine the p^{n-1} functions λ, in which the ratios $\lambda_1 : \lambda_2 : \cdots : \lambda_m$ are fixed, into one symbol $\xi\lambda\xi$. We obtain for $m > 2$ a transitive representation of the orthogonal quotient group. According as m is odd or even, this minimum number of letters is $(p^{n(m-1)}-1)/(p^n-1)$ or

$$(p^{nm/2} - \varepsilon)(p^{n(m/2-1)} + \varepsilon)/(p^n - 1)$$

$$[\varepsilon \equiv (-1)^{m(p^n-1)/4}]$$

For $m = 3$, the number is $p^n + 1$, in accord with the known isomorphism of the orthogonal quotient group with the linear fractional group in one variable.

These methods are then applied to the representation of every linear group defined by a quadratic invariant. The first hypoabelian group on $2m$ indices in the $GF[2^n]$ is represented as a transitive substitution group on either of the following numbers of letters:

$$(2^{nm}-1)(2^{n(m-1)}+1)/(2^n-1), \quad (2^{nm}-1)2^{n(m-1)},$$

the second number being larger than the first if $n > 1$ and smaller if $n = 1$. For the case $n = 1$, the small number is $2^{2m-1} - 2^{m-1}$, in accord with Jordan's proof of the isomorphism of the first hypoabelian group with the groups studied by Steiner in dealing with problems of contact of curves of the m^{th} order The second hypoabelian group is represented as a transitive substitution group on $(2^{nm}+1)(2^{n(m-1)}-1)/(2^n-1)$ letters.

In a paper to be presented to the *Mathematische Annalen*, the method is applied to the hyperorthogonal groups with the invariant $\sum_{i=1}^{m}\xi_i\bar{\xi}_i$.

Dr. Miller's paper deals in the first place with the group of isomorphisms of the abelian group H of type $(m-2, 1)$. This study is facilitated by the use of two theorems (proved in the introduction), viz.: The only operators that transform each subgroup of an abelian group into itself are those which transform each operator of the group into the same power, and the number of invariant operators in the group of isomorphisms of an abelian group is equal to the number of natural numbers which do not exceed the highest order of operators of the group and are prime to this order.

The group of isomorphisms of H contains p^3-1 operators of order p (when p is odd) forming, with identity, the non-abelian group of order p^3. When $p=2$ there are 15 operators of order 2 in the group of isomorphisms of H. These generate a group of order 32 containing 16 operators of order four. When $p>3$ there are seven non-abelian groups of order p^m which contain the abelian group of type $(m-2, 1)$; when $p=3$ there are eight such groups and the number of these groups is seventeen when $p=2$. The paper will be published in the *Transactions*.

Mr. Moritz's first paper, which will be published in the *American Journal of Mathematics*, is an extension of the ideas presented in a paper by the same author, of which an abstract is found on page 185, volume 6, of the BULLETIN. Starting with an abstract conception of an indeterminate, he arrives at the conclusion that the process of differentiation does not occupy the unique position which it is commonly supposed to occupy, but that there are other limiting processes. The equation $\lim\limits_{\lambda=1}\log_\lambda \dfrac{F(x^\lambda)}{F(x)} = \dfrac{gy}{gx}$ defines such a process, to which the term *quotientiation* is applied, $\dfrac{gy}{gx}$ being the quotiential coefficient of $y=F(x)$ with respect to x. The laws of this process are examined and the various rules for its application developed. It is suggested that a quotiential in which quotientiation takes the place of differentiation in the ordinary calculus, is not only conceivable, but might be applied, though with less ease than the differential calculus, to the investigation and interpretation of natural phenomena.

A further attempt at setting up other forms fails apparently, owing to the unsymmetric nature of the process a^b. If however $a^{\log b}$ is looked upon as derived by successive multiplication of a, just as $a \times b$ is obtained by successive additions of a, the difficulty vanishes completely. In fact, using De Morgan's form of the algebraic processes in which the process of the nth order is defined by $\log^{-n}(\log^n a + \log^n b)$, its inverse by $\log^{-n}(\log^n a - \log^n b)$, where n may be negative as well as positive, and where the 0th and 1st process are respectively identical with the ordinary addition and multiplication process, the writer finds that for every pair of consecutive processes there exists a limiting process, which when applied to any function $y = F(x)$ gives rise to a function $\dfrac{d_n y}{d_n x} = \log^{-n}\left(\dfrac{d \log^n y}{d \log^n x}\right)$, where n is the lower order of the processes involved. $\dfrac{d_n y}{d_n x}$ is called the *ratient* in the nth process of y with respect to x. Since n may be negative as well as positive, the chain of ratients can be extended in either direction without limit, the zero ratients being the ordinary differential coefficients. It is shown how any ratient may be expressed in terms of the ratient of any higher or lower order process. Successive ratients assume the form $\dfrac{d_n^r y}{d_n x^r} = \log^{-n} \dfrac{d^r \log^n y}{(d \log^n x)^r}$.

Mr. Moritz's second paper is based upon the definition of De Morgan's processes, and that of ratients as developed by the author in his paper on "Generalization of the differentiation process." A formula is developed for the ratient in the nth process of two or more functions combined by the mth process, from which the differentiation formulæ for a product of functions, powers of functions, or generally of functions combined by the process of any order, positive or negative, follow among others as simple corollaries. It is then shown that the fundamental theorems regarding the differentiation of a sum or a product hold for ratients in general when sums and products are replaced respectively by operands in the nth and $(n + 1)$th processes.

The number system for the nth process is examined and it is found that every De Morgan process possesses a distinctive number system whose properties are identical with those of the ordinary number system. To the imaginary i in the ordinary algebra correspond new imaginaries, to the transcendentals π and e correspond new transcendentals. In fact the transcendentals corresponding to π and e in the nth process are algebraic numbers borrowed from the $(n-1)$th and $(n+1)$th processes respectively. The totality of numbers may thus be conceived as arranged in an infinite number of strata, the ordinary algebraic numbers forming a single stratum.

With the extension of the differentiation process, and the stratification of the number body, all the results of the ordinary calculus, for example Taylor's theorem, can be expressed in terms of any two consecutive processes.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL.

THE VALUE OF $\int_0^{\pi/2} (\log 2 \cos \varphi)^m \varphi^n d\varphi$.

BY PROFESSOR F. MORLEY.

(Read before the American Mathematical Society, April 29, 1899.)

THE integral in question, in which m is any positive integer, and n is any even positive integer or zero, is given in effect, for the cases $m = 1$, $n = 0$, in Williamson's Integral Calculus, § 118 of the second edition, being taken from a paper by H. G. (presumably Harvey Goodwin, Bishop of Carlisle) in volume 3 of the *Quarterly Journal of Mathematics.* Further it is given in effect, for the case $m = 2$, $n = 0$, in Wolstenholme's Problems, p. 332 of the second edition. It seems worth while to show how it can be expressed in general, in terms of the constants $s_r = \sum_1^\infty 1/n^r$, which may be regarded as known.

We know that when p and q are real

$$1 + \frac{p\,q}{1\cdot 1} + \frac{p(p-1)q(q-1)}{1\cdot 2\cdot 1\cdot 2} + \cdots = \frac{\Gamma(1+p+q)}{\Gamma(1+p)\Gamma(1+q)},$$

when $p + q > -1$ (Forsyth, Differential equations, p. 197). But the left member is the constant term in the product of the series

$$(1+x)^p = 1 + px + \cdots$$

and

$$(1+1/x)^q = 1 + q/x + \cdots.$$

We suppose p and q positive, so that both series are absolutely convergent when $|x| = 1$. We may then multiply the series and get a Laurent series for

$$(1 + x)^p(1 + 1/x)^q.$$

Hence

$$\frac{1}{2i\pi}\int_{|x|=1}(1 + x)^p(1 + 1/x)^q\frac{dx}{x} = \frac{\Gamma(1 + p + q)}{\Gamma(1 + p)\Gamma(1 + q)}.$$

The path of integration shall be from -1 positively round the circle; the point -1 being a branch point of the integral, but not an infinity.

Writing $x = \exp 2i\varphi$, the integral becomes

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(2 \cos \varphi)^{p+q} \exp i(p - q)\varphi d\varphi.$$

Let

$$p + q = 2a, \quad p - q = 2\beta;$$

then

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\exp\{2a \log (2 \cos \varphi) + 2i\beta\varphi\}d\varphi$$

$$= \frac{\Gamma(1 + 2a)}{\Gamma(1 + a + \beta)\Gamma(1 + a - \beta)}.$$

Both sides can be developed in the form

$$\Sigma\Sigma C_{m,n}a^m\beta^n;$$

the left directly by the exponential theorem, giving

$$C_{m,n} = \frac{2^{m+n}}{\pi m!\ n!}\int_{-\pi/2}^{\pi/2}(\log 2 \cos \varphi)^m(i\varphi)^n d\varphi,$$

while the right, obtained from the formula

$$\log \Gamma(1 + n) = -\gamma n + \sum_2^\infty(-)^r\frac{s_r}{r}x^r,$$

is

$$\exp\left[\sum_2^\infty(-)^r\frac{s_r}{r}\{(2a)^r - (a + \beta)^r - (a - \beta)^r\}\right].$$

Hence the result: The integral

$$\frac{2}{\pi}\int_0^{\pi/2}(\log 2 \cos \varphi)^m\varphi^n d\varphi$$

where m is any positive integer, n any positive even integer or 0, is

$$(-)^{n/2}\frac{m!\,n!}{2^{m+n}}\times \text{the coefficient of } \alpha^m\beta^n \text{ in}$$

$$\exp\left[\sum_2^\infty (-)^r\frac{s_r}{r}\{(2\alpha)^r-(\alpha+\beta)^r-(\alpha-\beta)^r\}\right].$$

This generating function is, to a few terms,

$$\exp\,[s_2(\alpha^2-\beta^2)-2\,\alpha s_3(\alpha^2-\beta^2)$$
$$+\tfrac{1}{2}s_4\,(7\alpha^4-6\alpha^2\beta^2-\beta^4)-2\,\alpha s_5(3\alpha^4-2\alpha^2\beta^2-\beta^4)+\,]$$

or

$$1+s_2(\alpha^2-\beta^2)-2s_3(\alpha^3-\alpha\beta^2)$$
$$+\frac{s_2{}^2+7s_4}{2}\alpha^4-(s_2{}^2+3s_4)\alpha^2\beta^2+\frac{s_2{}^2-s_4}{2}\beta^4$$
$$-2(s_2s_3+3s_5)\alpha^5+4(s_2s_3+s_5)\alpha^3\beta^2-2(s_2s_3-s_5)\alpha\beta^4+\cdots$$

Thus, in tabular form, a few values for

$$\frac{2}{\pi}\int_0^{\pi/2}(\log 2\cos\varphi)^m\varphi^n d\varphi$$

are :

$n=$	0	2	4
$m=0$*	1	$\frac{1}{2}s_2$	$\frac{3}{4}(s_2{}^2-s_4)$
1	0	$-\frac{1}{2}s_3$	$-\frac{3}{2}(s_2s_3-s_5)$
2	$\frac{1}{2}s_2$	$\frac{1}{4}(s_2{}^2+3s_4)$	
3	$-\frac{3}{2}s_3$	$-\frac{3}{2}(s_2s_3+s_5)$	
4	$\frac{3}{4}(s_2{}^2+7s_4)$		
5	$-\frac{15}{2}(s_2s_3+3s_5)$		

ON THE ALGEBRAIC POTENTIAL CURVES.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, February 23, 1901.)

THE object of this paper is to derive the characteristic geometric properties of a class of curves which are of in-

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* The row for which $m=0$ is of course merely a verification, leading to the known values

$$s_2=\pi^2/6,\quad s_4=\pi^4/90.$$

terest in connection with the theory of equations and of the potential function. Analytically, these curves are obtained by equating to zero the rational integral solutions $\varphi(x, y)$ of Laplace's equation

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0,$$

or, what is equivalent, the real (or imaginary) parts of the rational integral functions of $x + iy$. Various geometric properties are given in Briot and Bouquet's Théorie des fonctions elliptiques (book IV, chapter II), but none are completely characteristic.

§ 1. *Apolarity with Respect to a Point Pair.*

A curve

$$\Phi \equiv a_x^n = 0,$$

is said to be apolar* to a conic

$$Q \equiv \Sigma P_{ik} u_i u_k \equiv u_P^2 = 0,$$

when every polar conic of the curve is circumscribed about an infinite number of triangles self-conjugate with respect to the conic Q; *i. e.*, when the bilinear covariant

$$S \equiv \Sigma P_{ik} \frac{\partial^2\Phi}{\partial x_i \partial x_k} \equiv a_P^2 a_x^{n-2} \tag{1}$$

vanishes identically.

Let the conic degenerate into a pair of points A, B,

$$A \equiv u_\alpha = 0, \quad B \equiv u_\beta = 0,$$

so that

$$Q = u_\alpha u_\beta;$$

then

$$S = a_\alpha a_\beta a_x^{n-2},$$

which is the apolar covariant of the forms

$$u_\alpha \quad \text{and} \quad u_\beta a_x^{n-1},$$

or of

$$u_\beta \quad \text{and} \quad u_\alpha a_x^{n-1}.$$

It is easy to show, however, that a point and a curve can be apolar only when the curve consists of a set of straight lines passing through the point, so that

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* Reye, *Crelle*, vol. 79 (1874), p. 159. For a convenient summary of the theory of apolar relations see Schlesinger, *Math. Annalen*, vol. 22 (1883), pp. 520–523.

THEOREM I. *If a curve of the nth order is apolar to a point pair (considered as a degenerate conic), the first polar (and therefore any of the polars) of either of the points with respect to the curve consists of a set of lines passing through the other point; the converse is also true.*

Letting $n = 2$, it follows that a conic is apolar to a point pair when the two points are conjugate with respect to the conic. From the definition of apolarity we have then

THEOREM II. *If a curve is apolar to a point pair, the latter is self-conjugate with respect to all the polar conics of the curve; conversely, etc.*

Since from Theorem I both the first polar and the polar conic of either point have nodes, we have

THEOREM III. *If a curve is apolar to a point pair, both the Hessian and the Steinerian of the curve pass through the point pair; furthermore, these points correspond in the sense defined by Clebsch.*[*]

§ 2. *Polar Properties of Potential Curves.*

Instead of an arbitrary point pair, consider now the pair of circular points at infinity I, J. The equation of this point pair in rectangular line coördinates may be written

$$u^2 + v^2 = 0;$$

so that, expressed in rectangular point coördinates, the covariant S of the preceding section becomes

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}.$$

The vanishing of this expression, however, denotes that the curve $\varphi = 0$ is a potential curve. Therefore,

THEOREM IV. *Any potential curve is apolar to the fundamental conic of euclidean geometry consisting of the circular points at infinity; conversely, any curve which is apolar to this fundamental conic is a potential curve.*[†]

From Theorem I we have then

THEOREM V. *All the polar curves of a circular point with respect to a potential curve degenerate into sets of straight lines passing through the other circular point; conversely, etc.*

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[*] Clebsch, "Ueber einige von Steiner behandelte Kurven," *Crelle*, vol. 64, p. 288. The converse of the above theorem is not true.

[†] Cf. Clifford, "On the canonical form of spherical harmonics," Works, p. 234, for a statement concerning "nodal curves" on a sphere, which appears to have some connection with the above.

A conic is a potential curve when the circular points I, J are conjugate with respect to it; this implies that the conic intersects the line at infinity in points which are harmonic with respect to I and J, *i. e.*, that the asymptotes of the conic are rectangular. From Theorem II, we have then a statement of the characteristic property of potential curves which has the advantage of dealing only with real elements, as follows:

THEOREM VI. *A curve is a potential curve when, and only when, the polar conics of all points with respect to the curve are rectangular hyperbolas.*

From Theorem III we have a property, which is however not characteristic, *i. e.*, not restricted to the potential curves:

THEOREM VII. *The Hessian and the Steinerian curves of a potential curve pass through the circular points* I, J; *furthermore, these points correspond in the sense defined by Clebsch.*

Since the polar conics of a polar curve are the polar conics of the original curve, we have

THEOREM VIII. *All the polar curves of a potential curve are themselves potential curves.*

§ 3. *Focal Properties of Potential Curves.*

Consider any rational integral function of the nth order in z

$$f(x+iy) \equiv \varphi(x, y) + i\psi(x, y), \tag{2}$$

together with the conjugate expression

$$\bar{f}(x-iy) \equiv \varphi(x, y) - i\psi(x, y); \tag{3}$$

the equation of the potential curve $\varphi = 0$ may be written in the form

$$f(x+iy) + \bar{f}(x-iy) = 0. \tag{4}$$

The two terms of the left hand member of this equation, equated separately to zero, represent sets of minimal lines, the first representing n lines through I, and the second n lines through J. Furthermore, equation (4) is unchanged when $f(z)$ is replaced by $f(z) + i\lambda$, where λ is an arbitrary real constant. We have then

THEOREM IX. *The linear system of curves of the nth order determined by $2n$ miminal lines (n through each of the circular points) is composed of potential curves; conversely, any potential curve may be obtained as a member of an infinite number of such linear systems.*

This theorem may be restated by using the fact that a curve of the nth class has n^2 foci, namely the intersections of the n minimal tangents of one system with the n minimal tangents of the other system, as follows:

THEOREM X. *Any curve of the nth order passing through the n^2 foci of a curve of the nth class is a potential curve; conversely, all potential curves may be obtained in this way—each potential curve passes through the foci of an infinite number of systems of confocal curves of the nth class.*[*]

Thus for $n = 2$, we have that all the conics which pass through the two real and the two imaginary foci of a conic are rectangular hyperbolas.

The potential curves $\varphi = 0$, $\psi = 0$, obtained in the decomposition of a function of $x + iy$, may be termed *conjugate*[†] *potential curves*, since the functions φ and ψ are conjugate. From (2) and (3) we have

$$(5) \qquad \begin{aligned} 2\varphi &= f(x+iy) + \bar{f}(x-iy), \\ 2i\psi &= f(x-iy) - \bar{f}(x-iy)\,; \end{aligned}$$

therefore the curves belong to a linear system of the kind considered above. Furthermore they intersect orthogonally.[‡]

THEOREM XI. *Conjugate potential curves of the nth order intersect orthogonally in the foci of a system of confocal curves of the nth class; conversely, two curves of the nth order which intersect orthogonally in the foci of a curve of the nth class are conjugate potential curves.*

The properties stated in Theorems IV and X being definitive for the same class of curves, it follows that these properties are equivalent. From this equivalence we may pass to a more general result relating to the apolarity of a curve and a point pair; it is necessary merely to project the circular points into an arbitrary pair of points, the potential curves transforming into curves which are apolar to this pair. Therefore, if through each of two points A, B, n straight lines are drawn, any curve of the nth order passing through the n^2 points so determined is apolar to the pair A, B; moreover this construction yields all the apolar curves. This result may be restated:

THEOREM XII. *A curve of the nth order is apolar to a pair of points, A, B when, and only when, it is possible to find upon*

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[*] The number of parameters in such a confocal system is $\frac{1}{2}n(n-1)$; so that the number of curves from which any potential curve may be derived as a focal curve is $\infty^{\frac{1}{2}(n^2-n+2)}$.

[†] The term conjugate, of course, here refers to the properties of the functions ϕ, ψ, and is not synonymous with the term apolar.

[‡] Briot et Bouquet, p. 223.

*the curve n^2 points lying by n's upon n lines through A and at the same time upon n lines through B.**

§ 4. *The Asymptotes.*

Briot and Bouquet prove† that the n asymptotes of a potential curve of the nth order are real, concurrent, and disposed symmetrically about their common point, the angle between consecutive asymptotes being $2\pi/n$. This property, however, imposes only $2n - 3$ independent conditions, while the number imposed by the equation‡ $\triangle\varphi \equiv 0$ is $\frac{1}{2}n(n-1)$; so that in general (*i. e.*, if $n > 3$) the above relation between the asymptotes is not peculiar to the potential curves, and Briot and Bouquet's theorem cannot be converted. As to the case $n = 2$ §, it has been shown in § 2 that the potential conics are the rectangular hyperbolas, so that the relation between the asymptotes is characteristic. The same is true in the case $n = 3$, as may be shown by taking a coördinate system with its origin at the point of concurrence of the asymptotes and its axis of abscissas coinciding with one of the asymptotes, and verifying the condition $\triangle\varphi = 0$. Therefore, *the potential cubics may be defined as those cubics which have three real concurrent asymptotes intersecting at angles of 120°.*

In all cases the *point of concurrence O of the asymptotes is a center* of the curve, *i. e.*, if any line is drawn through O, the sum of the distances measured from O of the points of intersection lying on one side of O is the same as the corresponding sum for the points on the other side. This follows from the fact that when the origin of coördinates is taken at O, all the terms of order $n - 1$ disappear. ||

From the potential curves we may pass by projection to the curves which are apolar to any point pair A, B. The asymptotes of the potential curve are transformed into a set of concurrent lines tangent to the new curve at the points $P_1, P_2, \cdots, P_n$ where the line joining A, B cuts the curve. From the equality of the angles between consecutive asymptotes, the anharmonic ratios

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* When one such set of n^2 points exists there is necessarily an infinite number of sets.

† L. c., p. 227.

‡ The number of parameters in the potential curve of the nth order is $2n$, so that of all the curves of the nth order which pass through $2n$ assigned points one and only one is potential.

§ The case $n = 1$ is trivial since all straight lines are potential curves.

|| Briot et Bouquet, p. 226; Salmon-Fiedler, Hohere Kurven, 2d ed. p. 145.

$$(A\ B\ P_1\ P_2),\quad (A\ B\ P_2\ P_3),\quad \cdots,\quad (A\ B\ P_n\ P_1)$$

are equal, and therefore the set of points $P_1, \cdots, P_n$ is apolar to the pair A, B.

THEOREM XIII. *If a curve is apolar to a point pair, the line through the pair intersects the curve in a set of points apolar to the pair, and the tangents to the curve at these points are concurrent.*

The converse is true only for conics and cubics.

§ 4. *Connection with the Theory of Equations.*

Consider the general equation of the nth degree in one unknown

$$(6)\qquad f(z) \equiv A_0 z^n + A_1 z^{n-1} + \cdots + A_n = 0 \qquad (A_k = b_k + ic_k),$$

with the n roots

$$z_\alpha = x_\alpha + iy_\alpha \qquad (\alpha = 1, 2, \cdots, n).$$

The conjugate equation

$$\bar{f}(z) \equiv \bar{A}_0 z^n + \bar{A}_1 z^{n-1} + \cdots + \bar{A}_n = 0 \qquad (\bar{A}_k = b_k - ic_k),$$

then has the roots

$$(7)\qquad \bar{z}_\alpha = x_\alpha - iy_\alpha \qquad (\alpha = 1, 2, \cdots, n).$$

The complete solution of equation (6) is equivalent directly to the real solution of the system

$$(8)\qquad \varphi(x, y) = 0, \qquad \psi(x, y) = 0,$$

where φ and ψ are the real and imaginary parts of $f(x+iy)$. A problem which then presents itself, namely, the complete solution of this system, is virtually answered in Theorem X: the solutions, by (5), are obtained by solving the linear equations

$$x + iy = z_\alpha, \quad x - iy = \bar{z}_\beta \qquad (\alpha, \beta = 1, 2, \cdots, n).$$

THEOREM XIV. *The complete solution of the auxiliary system* (8) *connected with the equation* (6) *is*

$$x_{\alpha\beta} = \frac{z_\alpha + \bar{z}_\beta}{2}, \quad y_{\alpha\beta} = \frac{z_\alpha - \bar{z}_\beta}{2i} \qquad (\alpha, \beta = 1, 2, \cdots, n).$$

The n real solutions x_α, y_α are obtained by letting $\beta = \alpha$; the remaining solutions may be expressed in terms of these as follows:

$$x_{\alpha\beta} = \frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad y_{\alpha\beta} = \frac{y_\alpha + y_\beta}{2} - i\frac{x_\alpha - x_\beta}{2}.$$

As is well known, the n quantities

$$R(z_\alpha) \qquad (\alpha = 1, 2, \cdots, n),$$

where R denotes any rational function, satisfy an equation of the nth degree

$$F_n(t) = 0,$$

whose coefficients are rational in A_0, $A_1, \cdots, A_n$. This, however, no longer holds when we consider, instead of rational functions of the roots, rational functions of the real and imaginary parts of the roots; but if we consider the n^2 quantities

$$R(x_{\alpha\beta}, y_{\alpha\beta}),$$

they will satisfy an equation of the n^2 degree with coefficients which are rational in terms of the coefficients of φ and ψ, *i. e.*, in terms of $b_0, \cdots, b_n, c_0, \cdots, c_n$. Therefore,

Theorem XV. *The n quantities*

$$R(x_1, y_1), \quad R(x_2, y_2), \cdots, R(x_n, y_n),$$

are the real roots of an equation of degree n^2 with coefficients which are rational in terms of the real and imaginary parts of the coefficients in (1) ; *the remaining roots of the equation being*

$$R\left(\frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad \frac{y_\alpha + y_\beta}{2} + i\frac{x_\alpha - x_\beta}{2}\right), \quad (\alpha \neq \beta).$$

This result may easily be extended to functions of any number of roots $R(x_1, y_1, x_2, y_2, \cdots)$, and Theorem XIV may be extended to any system of simultaneous equations.

Columbia University,
February 25, 1901.

ALTERNATING CURRENT PHENOMENA.

Alternating Current Phenomena. By C. P. Steinmetz. New York, Office of the Electrical World. Third Edition, 1900. Pp. xx + 525.

To electrical engineers Mr. Steinmetz's book is immediately conspicuous by reason of two distinguishing characteristics :

the first is the employment of a definite mathematical method of presentation, consistently maintained throughout the course of the work, and the second the employment of this method in the analysis of practically every problem in the application of alternating currents of electricity. A glance over the literature of applied electricity reveals no other work which stands forth so prominently in either of these characteristics, and the value of a treatment embracing them both can only be rightly estimated by those who have worked out their basic conceptions of alternating current phenomena and their applications from the conglomerate mass of trigonometry, differential equations, and inaccurate diagrams presented by earlier writers, and, found how inadequate it is for the solution of the problems confronting the engineer of today.

It is the first of these characteristics, namely the method of treatment that is the more interesting to mathematicians, and it is the purpose of this article to review the application of this method; a critical discussion of the complete work from the standpoint of the electrical engineer is not aimed at, and so all reference is omitted to much of the matter that is most valuable but hardly of interest in this place. Briefly stated, the method is the use of the algebra of the complex number in combination with a reference to polar coördinates of the alternating or periodic functions of current and electromotive force. A short consideration of a simple electric circuit carrying an alternating current will facilitate a review of the use to which this method has been put.

Threading or looping with a circuit carrying a current, there is a number of lines of magnetic force due to the passage of the current around the circuit; and this number rises, falls, and reverses with the varying values of the current; the induction or total number of these lines is thus also a periodically varying function "in phase" with the current. Due to this alternating field of force there is induced in the circuit an alternating electromotive force, which is shown by the law of Lenz to have its maximum one-quarter of the time of one complete period later than the inducing field, and so 90° later than that of the current, 360° representing a complete period; this is the counter electromotive force of self-induction. Due to the resistance of the conductor there is a consumption of electromotive force when the current flows, proportional to the current at each instant and so alternating and in phase with the current; this may be considered a counter electromotive force, 180° away from the current. If there is a condenser or electrical capacity in the circuit there is a third electromotive force which may be shown to have its phase 90° in advance of the current. It is the presence of these several angularly separated electromotive forces which causes the apparent failure of Ohm's law in the case of alternating current circuits. The impressed electromotive force necessary to cause the current to flow must overcome these several electromotive forces, *i. e.*, must have components angularly separated, and so in general will not be in phase with the current. It is to be noted that the difference in phase between current and counter electromotive force is either 0° or plus or minus 90°.

The fundamental principles suggested in the foregoing paragraph are assumed in the opening chapter of Mr. Steinmetz's book, as is also the form of the expression giving the value of the "impedance" or ratio of impressed E.M.F. to current. This expression for the impedance is $z = \sqrt{r^2 + x^2}$, r being the resistance or ratio of the in-phase component of the E.M.F. to the total current, and x the "reactance" or ratio of the out-of-phase component of the E.M.F. to the total current; since as indicated above there are two out-of-phase E.M.F.'s, one 90° in advance of, the other lagging 90° behind the current, x will take its value from the difference between the two E.M.F.'s due to self-induction and capacity, since they differ in phase by 180° or are opposed to each other. The values of x for self-induction and for capacity are calculated in terms of the frequency or number of periods per second and the constants of the circuit.

Passing now to chapter IV, the alternating or sinusoidal wave, represented by time as abscissa and instantaneous value as ordinate, is referred to polar coördinates, giving the circle as the curve; and for each complete period the circle is traversed twice, negative values of the function being taken when a reverse direction must be taken by the radius vector in order to intersect the curve. Thus the intercept on any radius vector gives the instantaneous value of the wave at the time represented by the amplitude of the vector. Since any particular value determines the wave, the step is then made of letting the diameter of the characteristic circle represent the wave, by its length the intensity and by its amplitude the phase; thus on the same diagram any number of E.M.F.'s and currents in a circuit differing in intensity and in phase may be represented by radii vectors of different lengths and amplitudes. The author here tacitly assumes that the current has at any instant the same value throughout the whole length of the circuit, *i. e.*, that the phase of the current wave does not change from point to point; cases in which this is unwarranted are considered in chapter XIII. The possibility of combining or resolving vectors of the same nature by the parallelogram law is then shown by considering combined instantaneous values on any chosen radius vector; thus the resultant of two electromotive forces. for instance, is represented by the diagonal of the parallelogram formed on the two radii representing their intensities and phases. The graphical method here clearly developed gives perhaps the clearest insight possible into the mutual relations of the several alternating sine waves entering into any problem.

Owing to the widely differing magnitudes of the alternating waves to be represented in the same diagram, the graphical method is not well suited for numerical calculation, and in chapter V the author extends the graphical treatment into the symbolic method, which, instead of denoting the vector representing the sine wave by the polar coördinates of intensity I and amplitude ω, uses the rectangular coördinates $a = I \cos \omega$ and $b = I \sin \omega$, thus avoiding the use of trigonometric functions in the combination or resolution of waves. Extending the relations of the parallelogram law: "Sine waves are combined or resolved by adding or subtracting their rectangular components." To distinguish between the two components, the symbol j is put before the vertical component, $I = a + jb$, meaning that a is the horizontal and b the vertical component of the wave I and that they are combined in the wave of resultant intensity $i = \sqrt{a^2 + b^2}$; similarly $a - jb$ is a wave with a as horizontal and $-b$ as vertical component. The next step brings in the full significance of the method; multiplying the symbolic expression $a + jb$ by -1 evidently gives $-a - jb$, or a wave of equal intensity but differing in phase by 180°; a wave of equal intensity but lagging in phase by 90° (clockwise rotation) is evidently represented by $ja - b$; this expression may be derived by multiplying the expression $a + jb$ by j if upon the "hitherto meaningless symbol j" the condition be imposed $j^2 = -1$; and similarly multiplying by $-j$ advances the wave through 90° or one quarter of a period. The symbol j is thus seen to be the imaginary unit, and the sine wave is represented by the complex quantity of the type $a + jb$; the letter j is used instead of the usual i, since the latter so commonly in electrical literature denotes the current. "As the imaginary unit j has no meaning in the system of ordinary numbers, this definition of $j = \sqrt{-1}$ does not contradict its original introduction as a meaningless symbol." Thus $a + jb$ means a wave of intensity $i = \sqrt{a^2 + b^2}$ and of phase $\omega = \tan^{-}{}_{1}\frac{b}{a}$; it may also be represented by $i(\cos\omega + j\sin\omega)$ and also by $i\varepsilon^{j\omega}$. A further extension of method now gives: "Sine waves may be combined or resolved by adding or subtracting their complex algebraic expressions." The complex expression for the impedance is then developed and may here serve as a simple instance of the use of the method. A current $I = i + ji'$ flows in a circuit of resistance r and reactance x; the E.M.F. consumed by resistance is in phase with the current and is $rI = ri + jri'$; this E.M.F. must be supplied by the impressed E.M.F., as must also be an E.M.F. necessary to overcome the counter E.M.F. due to the reactance x; this E.M.F., if it be due to self induction, lags 90° behind the current and is therefore represented by $ixI = jxi - xi'$ (if x be due to capacity, by $-jxI = xi' - jxi$); the component of the impressed E.M.F. to overcome this is evidently $-jxI = -jxi + xi'$; and the E.M.F. to overcome both resistance r, and reactance x is $E = (r - jx)\,I$; or, the ratio of electromotive force to current, *i. e.*, the impedance, has for its complex expression $Z = r - jx$. The relation $E = ZI$, the complex values of the three quantities being used, may be handled in any of its three forms, giving a simple complex expression for any one of the quantities in terms of the components of the other two, since the imaginary may be easily eliminated from a denominator, and thus the real and imaginary components separated.

From the foregoing, it is seen that the total impedance of a circuit, consisting of any number of portions differing as to r and x, and connected in series, may be obtained by adding the complex expressions for the impedance of the several portions. The total impedance being known, the E.M.F. necessary to supply any given value of the current to such a circuit may at once be had, together with its phase relation to the current. This is the general nature of one class of problems. If, however, the circuit has several branches, or if several currents are supplied by the same E.M.F., the total impedance of the circuit is not a simple expression, just as the resistance of a number of branches connected in parallel is not a simple expression in the resistances of the several branches. In the latter case, however, the joint conducting power, *i. e.*, the reciprocal of the resistance, is the sum of the "conductances" of the several branches. So for parallel connected branches the author derives in chapter VII a method for combining, by the addition of complex quantities, the effects of the several impedances of a branched circuit. The total current supplied by the impressed E. M.F. is the sum of the currents in the several branches, attention being paid to their phase relations, *i. e.*, the complex expressions of the currents are to be added. To simplify this, Ohm's law, which now holds if complex expressions are used, is put in the form $I = EY$, where Y, being the reciprocal of Z, is a complex quantity. and the values of I for the several branches are now readily added. Y is called the admittance, and the total admittance of a branched circuit is the sum of the complex expressions of the individual admittances. Y, being complex, is of the form $g + jb$. We have

$$Y = g + jb = \frac{1}{Z} = \frac{1}{r - jx} = \frac{r + jx}{r^2 + x^2}.$$

Therefore

$$g = \frac{r}{r^2 + x^2} \quad \text{and} \quad b = \frac{x}{r^2 + x^2},$$

so that the expressions for Y and Z for any branch or combination of branches are readily derived one from the other.

Chapters VIII and IX give a complete investigation of the various types of series and parallel circuits, of the effects upon regulation and phase difference of the relative values of r, x, g, and b; and results heretofore obtained only in very complicated form are reduced to simple algebraic expressions. A single simple case will suffice here, but no electrical engineer should fail thoroughly to digest the contents of these two chapters.

A reactance x_0 is inserted in series with a load circuit of impedance $Z = r - jx$, and an E.M.F. E impressed upon the whole. The total impedance is $Z - jx_0$ or $r - j(x + x_0)$; the current is

$$I = \frac{E_0}{r - j(x + x_0)},$$

with the absolute value $\dfrac{E_0}{\sqrt{r^2 + (x + x_0)^2}} = \dfrac{E_0}{\sqrt{z^2 + 2xx_0 + x_0^2}}$;

the E.M.F. on the receiver or load circuit is

$$E = IZ = \frac{E_0(r - jx)}{r - j(x + x_0)} = E_0 \sqrt{\frac{r^2 + x^2}{r^2 + (x + x_0)^2}}$$

$$= \frac{E_0 z^2}{\sqrt{z^2 + 2xx_0 + x_0^2}}.$$

Generally in such a case the value of E, as compared with E_0, is of prime importance, and the above expression gives means of controlling E by a proper variation of x_0; thus $E = E_0$ if $x_0 = -2x$; if $x_0 < -2x$ it raises, if $x_0 > -2x$ it lowers the voltage; if x and x_0 have the same sign E is always less than E_0. It is to be remembered that a positive value of x is given by self-induction, a negative value by capacity. The difference in phase between current and E.M.F. is gotten, as indicated above, from the expression for the impedance; here the difference in phase in the load circuit is $\omega = \tan^{-1}\frac{x}{r}$; and in the supply or generator circuit $\omega' = \tan^{-1}\frac{x + x_0}{r}$.

Chapter XII is new in the third edition; in it the author attempts to extend the symbolic method to quantities of double frequency, such as the power. At any instant the flow of power in an alternating current circuit is the product of the instantaneous values of current and E.M.F. If two sine waves, *e. g.*, one of current and one of E.M.F. differing in phase, be drawn in rectangular coördinates, and if also a curve representing the product of their instantaneous values be drawn, it is found that while either the current or E.M.F. has passed through half a period, the curve of products, or the power wave, has passed through a complete period; that is, the power has double the frequency of the current and E.M.F., and so may not be represented on the same vector diagram with them. The area of positive values in the power curve represents power given into the medium by the circuit, that of negative values power returned to the circuit from the medium, this power having been stored there in the forms of a magnetic field and an electrostatic strain; the difference is the true expenditure of power. The product of the complex expressions for current and E.M.F., $(e' + je'')(i' + ji'') = (e'i' - e''i'') + j(e''i' + e'i'')$, does not represent the power, since it is an expression of the same frequency as the current and E.M.F.; suppose, however, since the power is of double frequency, the phase angle be doubled in the above expression; *i. e.*, instead of $j^2 = -1$, corresponding to a rotation through 180°, we now have $j^2 = +1$, or 360° rotation, and multiplication by j merely reverses the sign or rotates through 180°. The product then becomes $(e'i' + e''i'') + j(e''i' - e'i'')$, the first term of which is the real power $EI \cos \omega$, and the second the author calls the "wattless power," or $EI \sin \omega$. The chapter is interesting in explaining the apparent failure of the product of the two complex expressions to represent the power, but is unattractive, except to those versed in non-commutative algebra, because of the necessity of remembering that $j \times 1 = j$ is not the same as $1 \times j = -j$.

Chapter XIII is devoted to those cases where it is not permissible to assume that at any instant the value of the current is the same throughout the circuit; an instance is a submarine cable or any line along which capacity in some quantity is uniformly distributed. In such cases the simple vector diagram and the algebra of complex quantities do not suffice; however, by considering the values of r, x, g, b per unit length of line, the author forms and solves the differential equations for both current and E.M.F., as varying from point to point in the line. While exhaustive in discussion and most useful, the chapter offers no striking application of the symbolic method.

In chapters XIV, XV, and XVI the symbolic method is extended to the analysis of the transformer and the induction motor, *i. e.*, the motor with rotating magnetic field; they are shown to belong to the same general type of apparatus (a fact not before recognized), called by the author the general alternating current transformer. Consider the simple transformer consisting of a magnetic circuit interlinked with two electric circuits, a primary and a secondary. The primary circuit carrying current sets up a field in the magnetic circuit, which induces an E.M.F. in the secondary which supplies current to its load. The secondary is now considered as a simple circuit with a given impressed E.M.F., having an internal impedance $Z_1 = r_1 - jx_1$ due to its resistance and self-induction, and feeding a load circuit of impedance $Z = r - jx$. Since the same magnetic circuit links with both coils, the actions in the secondary are shown to be reducible to the primary by the ratio of the numbers of turns in each; the primary has also its internal impedance $Z_0 = r_0 - jx_0$; combining the primary impedance with the reduced values of the secondary circuit, the effect of the whole transformer is brought to the expression of a single impedance. By proper attention to the difference in frequency between primary and secondary due to the "slip" of the armature (secondary) behind the rotating field due to the primary, the same method of procedure is adopted for the induction motor.

Chapter XXIV is a most interesting extension of the symbolic method to the representation of the general alternating wave as distinguished from the simple sine wave of the type $A = a_0 \cos (\varphi - \alpha)$; to the latter only is the vector representation $A = a' + ja'' = a(\cos \alpha + j \sin \alpha)$ applicable.

If the two half periods of a wave are similar, the even harmonics are absent and the general wave is expressed by

$$A = A_1 \cos(\varphi - a_1) + A_3 \cos(3\varphi - a_3) + A_5 \cos(5\varphi - a_5) + \cdots,$$

which may not be represented by a single complex vector quantity. The individual harmonics, however, of this general wave are independent and no products appear, so that each may be represented by a complex symbol and the symbolic expression for the general wave is

$$A = \sum_1^\infty (2n-1)(a_n' + j_n a_n'');$$

here $j_n = \sqrt{-1}$ always, but the index of j_n denotes that the j's of different indices, while equal algebraically, physically represent different frequencies and so cannot be combined. The general wave of E.M.F. is thus represented by

$$E = \sum_1^\infty (2n-1)(e_n' + j_n e_n''),$$

and the current by a similar expression in the i's. The expression for the impedance undergoes some change; the values of x, the reactance, when due to self-induction, are directly, when due to capacity, inversely proportional to the frequency; there is also a part independent of the frequency; thus the impedance of a circuit will have different values for the several harmonics and its general expression is

$$Z = r - j_n\left(nx_m + x_0 + \frac{x_c}{n}\right).$$

Operations according to Ohm's law may now be performed on the general wave quantities E, I, and Z, just as on the simple sine wave; multiplication and division, however, being only performed on those terms having the same index n.

A common and most useful medium for the use of alternating currents is the so called polyphase system in which several equal E.M.F.'s differing in phase by the same angle are generated in the same machine; the induction motor is the most conspicuous form of apparatus depending on the polyphase system. The symbolic method lends itself admirably to the representation of such a system by means of the nth roots of unity. In the polar diagram the n E.M.F.'s of an n-phase system are represented by n equal vectors following each other under equal angles $2/\pi n$. In symbolic notation, advance or rotation through an angle $2\pi/n$ is represented by multiplying by the quantity $\cos 2\pi/n + j \sin 2\pi/n$, and so the E.M.F.'s of a polyphase system are

$$E,\quad E\left(\cos\frac{2\pi}{n}+j\sin\frac{2\pi}{n}\right),\quad E\left(\cos\frac{4\pi}{n}+j\sin\frac{2\pi}{n}\right),\ \text{etc.}$$

In chapters XXVI and XXVIII the author handles this application for the deduction of the expression for the rotating magnetic field, the ring and star E.M.F.'s of interlinked systems, and other matters of general use, but up to this time wanting an analytical expression.

The writer has only gratitude to express at the appearance of this work, and his one regret is that its author did not also include in it his recent articles on the rotary converter.

JOHN B. WHITEHEAD, JR.

JOHNS HOPKINS UNIVERSITY,
April 18, 1901.

SHORTER NOTICE.

Leçons Nouvelles sur les Applications Géométriques du Calcul Différentiel. By W. DE TANNENBERG. Paris, A. Hermann, 1899. 192 pp.

THIS volume, which M. de Tannenberg has contributed to the literature of the theory of curves and surfaces, is very opportune. We have wanted a book which would make possible for the beginner a knowledge of the more fundamental geometrical applications of the calculus and in a way which would prepare him for the treatises of Darboux and Bianchi. To be sure, this field has been covered, more or less, in the chapters devoted to geometrical applications in the French treatises on analysis—notably by Jordan, Picard, Appell—but rather as examples of the methods of analysis and not standing forth as a systematic development of the elements of another field of mathematics. Again, there have been in recent years, quite a number of shorter treatises with just the scope of the volume under discussion, but their treatment of the subject has been along lines quite different from the well known methods of the calculus: Ricci in his Lezioni arrives at the results by the study of differential forms, Méray makes use of the methods which he has developed in his Leçons nouvelles sur l'analyse infinitésimale, Fehr adopts the vector method of Grassman.

But M. de Tannenberg has given us a treatment of the subject by methods which involve only the simplest notions of the calculus, and in a way that is at the same time clear, concise, and rigorous. Limiting himself to the study of curves and surfaces in the region of ordinary points, he considers the cartesian coördinates as developed according to ascending powers of one or two variables, and from the study of these developments determines many of the geometrical properties of the given locus. As one reads, he is impressed with the simplicity of the methods; very often an important result follows as a corollary, or is embodied in a remark, and at times the examples which are scattered through the book serve the double purpose of illustration and a step toward the further development of the subject. Very little is said about plane curves; and, when treated, it is simply as a particular case of curves in general.

The book has two main divisions: the first, consisting of two parts, treats of the descriptive properties of curves and surfaces; the second, in three parts, treats of their metric properties.

Of particular interest in the first division is the treatment of the theory of envelopes, whether of curves or surfaces. The conditions necessary and sufficient for the existence of envelopes are set forth so clearly that the beginner should not experience the usual difficulty with this subject. The second part closes with a brief discussion of systems of lines —ruled surfaces, congruences, and linear complexes.

In the third part of the book and the first of the second division, the notion of the geometrical derivative of a segment is introduced and is of great service in the definition of first curvature and torsion of a curve, and the derivation of the Serret formulæ. As an application of these formulas, the variation of a segment of a right line is discussed, and the results thus obtained enable the author to simplify the proofs of many theorems in the theory of surfaces. In particular, they are applied to the study of skew and developable surfaces; this forms the fourth part.

The fifth part is the most important; it comprises nearly one-half of the book and is devoted to the theory of curved surfaces. First the six fundamental functions, characteristic of a surface, are defined and the relations which they satisfy are found; then the general properties of curves traced on surfaces are discussed. From the theorems of

Meusnier and Bonnet follow the definitions of asymptotic lines and lines of curvature; their properties, together with those of conjugate lines, are then discussed and geodesic lines are defined. After an application of the preceding theories to several of the simpler surfaces—cones, surfaces of revolution, quadrics, developables, surfaces of Monge—the book closes with a treatment of the general equation of geodesic lines.

The print is good, the figures are helpful and the general arrangement is very attractive.

L. P. EISENHART.

NOTES.

THE secretaryship of Section A of the American association for the advancement of science has been filled by the appointment of Dr. G. A. MILLER. Papers intended for the programme of the Denver meeting (August 24–31) of Section A, should be sent to Dr. Miller, at 115 Cook St., Ithaca, N. Y.

THE April number of the *American Journal of Mathematics* (volume 23, number 2) contains the following articles: "The cross-ratio group of 120 quadratic Cremona transformations of the plane," by H. E. SLAUGHT; "Memoir on the algebra of symbolic logic," by A. N. WHITEHEAD; "On a special form of annular surfaces," by V. SNYDER; "On the transitive substitution groups whose order is a power of a prime number," by G. A. MILLER; "Geometry on the cubic scroll of the second kind," by F. C. FERRY.

AT the regular meeting of the London mathematical society held on May 9, 1901, the following papers were read: "A case of algebraic partitionment," by Major P. A. MACMAHON; "On the series whose terms are the cubes and higher powers of the binomial coefficients," by Major P. A. MACMAHON; "A property of recurring series," by Dr. G. B. MATHEWS; "The product of two spherical surface harmonic functions," by Mr. J. B. DALE.

THE German Society for the promotion of instruction in mathematics and science held its tenth annual meeting the last week of May at Giessen under the presidency of Dr. Rausch. The society has nine hundred members.

THE committee elected to judge the competition in 1901 for the Francoeur and Poncelet prizes of the Paris academy of sciences consists of Professors C. JORDAN, H. POINCARÉ, E. PICARD, P. APPELL, and MAURICE LEVY.

THE following additional volumes (see BULLETIN, volume 6, pp. 306 and 355) are announced as either in preparation or under the press for B. G. Teubner's series of text-books in the mathematical sciences: G. BOHLMANN, The mathematics of insurance; G. H. BRYAN, Thermodynamics; L. E. DICKSON, Linear groups, with an exposition of the Galois field, theory (English, in press); G. ENESTRÖM, with the coöperation of other scholars, Handbook on the history of mathematics; PH. FURTWÄNGLER, Mechanics of the simplest kinds of physical apparatus and of experiments; M. GRÜBLER, Hydraulic motors; L. HENNEBERG, Text-book of graphical statics; K. HEUN, The kinetic problems of the modern theory of machines; G. JUNG, Geometry of masses; H. LAMB, Acoustics; A. LOEWY, Lectures on the theory of linear substitution groups; A. E. H. LOVE, Textbook of hydrodynamics; A. E. H. LOVE, Theory of elasticity; W. MEYERHOFFER, The mathematical foundations of chemistry; E. OVAZZA (from the field of mechanics); FR. POCKELS, Optics of crystals; P. STÄCKEL, Textbook of general dynamics; H. E. TIMERDING, Theory of systems of vectors and screws; A. VOSS, Principles of rational mechanics. Of the volumes previously announced three have appeared, viz., E. PASCAL, Determinants; O. STOLZ and J. A. GMEINER, Theoretical arithmetic, Part I; E. v. WEBER, Lectures on the problem of Pfaff. Three others are in press, viz., L. E. DICKSON, Linear groups (see above); G. LORIA, The theory and history of special algebraic and transcendental plane curves; E. NETTO, Theory of combinations.

SUBSCRIPTIONS are invited toward the publication of the complete works of the late Professor E. BELTRAMI (see BULLETIN, volume 7, p. 46), which has been undertaken by the faculty of science of the University of Rome. There will be four large volumes with about 2000 pages in all. A copy of the complete works will be sent to those who subscribe not less than fifty lire. Subscriptions may be sent to ISAIA SONZOGNO, Piazza S. Pietro in Vincoli, 5, Rome.

IT appears, from *Nature*, that the English civil service commission no longer insists, in its examinations in geometry, on preserving the Euclidean sequence of propositions, but now accepts correct demonstrations whether those of Euclid or not. Recent papers also encourage teaching of a less abstract character than that usually associated with Euclid's geometry.

To the list of mathematical courses to be given at Columbia University in 1901–1902, as announced in the last number of the BULLETIN, there is to be added a course on "The history of mathematics," three hours weekly, by Professor D. E. SMITH.

A BUST of GAUSS has been placed in one of the halls of the University of Berlin.

PROFESSOR E. H. MOORE is one of the five new members elected to the National academy of sciences at its annual meeting April 16th–18th.

DR. J. A. GMEINER has been made professor extraordinarius of mathematics at the German University of Prague.

PROFESSOR GINO FANO, now at Messina, has been made assistant professor of projective and descriptive geometry at the University of Turin. Professor U. DINI of the University of Pisa has been elected a member of the superior council of instruction of Italy.

AT Yale University Professor J. E. CLARK, James E. English professor of mathematics in the Sheffield Scientific School since 1873, has resigned because of poor health. He has been made professor emeritus, and Professor P. F. SMITH has received the title of James E. English professor. Dr. E. R. HEDRICK has been appointed instructor in mathematics in the Sheffield Scientific School.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. W. R.). A short account of the history of mathematics. 3d edition. London and New York, Macmillan, 1901. 12mo. 552 pp. Cloth. 10s.

BEMAN (W. W.). See DEDEKIND (R.).

BLICHFELDT (H.). On a certain class of groups of transformations in space of three dimensions. (Diss.) Leipzig, 1900. 8vo. 61 pp.

BONNEL (J.). Note sur les systèmes de géométrie et l'atome. Paris, Gauthier-Villars, 1900. 8vo. 15 pp. Fr. 1.00

CADENAT (A.) See RUSSELL (B. A. W.).

CARNOY (J.). Cours d'algèbre supérieure : principes de la théorie des déterminants ; théorie des équations ; introduction à la théorie des formes algébriques. 2ᵉ édition. Paris, Gauthier-Villars, 1900. 8vo. Fr. 11.00

COUTURAT (L.). See RUSSELL (B. A. W.).

CRIVETZ (T.). Essai sur l'équidistante. Bucarest, Göbl, 1900. 8vo. 50 pp.

DARREYE (A.). Polare Felder und Kegelschnitte mit gemeinsamem Poldreieck. (Diss.) Strassburg, 1900. 8vo. 41 pp., 2 plates.

DAVISSON (S. C.). Ueber die geodätische Linie der Manigfaltigkeit $ds^2 = dx^2 + \sin^2 x\, dy + dz^2$. (Diss.) Tübingen, 1900. 8vo. 22 pp.

DEDEKIND (R.). Essays on the theory of numbers: (1) Continuity and irrational numbers; (2) The nature and meaning of numbers. Authorized translation by W. W. Beman. Chicago, Open Court Publishing Company, 1901. 12mo. 4 + 115 pp. Cloth. $0.75

DÜRLL (W.). Die Probleme des logarithmischen Potentials für eine von zwei Kreisbogen begrenzte ebene Fläche. (Diss.) Leipzig [1900]. 8vo. 134 pp., 2 plates.

ERMAKOF (V. P.). Lectures on analytic geometry delivered in the University and in the Polytechnic Institute. Part II: Solid geometry. Kief, 1900. 8vo. 208 pp (Russian.) M. 7.00

——. Lectures on the integral calculus. Parts I and II. Kief, 1900. 8vo. 350 pp. (Russian.) M. 9.00

GUTZMER (A.), HENSEL (K.), HEUN (K.). See JAHRESBERICHT.

HILBERT (D.). Les principes fondamentaux de la géométrie. Traduit par L. Laugel. Paris, Gauthier-Villars, 1900. 4to. 114 pp. Fr. 5.00

JACOBSTHAL (W.). Ueber die asymptotische Darstellung der Integrale einer gewissen linearen Differentialgleichung zweiter Ordnung. (Diss.) Strassburg, 1899. 8vo. 40 pp

JAHRESBERICHT der deutschen Mathematiker-Vereinigung. Vol 9: enthaltend die Chronik der Vereinigung für das Jahr 1900, die auf der Versammlung in Aachen gehaltenen Vortrage, sowie: Die kinetischen Probleme der wissenschaftlichen Technik, Bericht von K. Heun. Herausgegeben im Auftrage des Vorstandes von K. Hensel und A. Gutzmer. Leipzig, Teubner, 1901. 8vo. 4 + 140 + 6 + 123 pp. M. 8.00

JUNKER (F.). Höhere Analysis. Teil I: Differentialrechnung. 2te Auflage. Leipzig, Göschen, 1901. 12mo. 231 pp. Cloth. (Sammlung Goschen, No. 87.) M. 0.80

KARSTENS (H.). Ueber gewisse asymptotische Lösungen der Differentialgleichungen der analytischen Mechanik. (Diss.) Berlin, Mayer & Muller, 1901. 8vo. 37 pp. M. 1.20

KILLING (W.). Lehrbuch der analytischen Geometrie in homogenen Koordinaten. Teil 2: Die Geometrie des Raumes. Paderborn, Schoningh, 1901. 8vo. 8 + 361 pp. M. 5.60

LAUGEL (L.). See HILBERT (D.).

MARO (L.). Conforme Abbildung eines von irregulären Hyperbeln $n$ter Ordnung begrenzten Flächenstückes auf dem Einheitskreise. (Diss.) München, 1899. 8vo. 27 pp.

MARKOF (A. A.). Calculus of probabilities. St. Petersburg, 1900. 8vo. 279 pp. (Russian.)

MEYER (P.). Ueber die Siebenteilung der Lemniskate. (Diss.) Strassburg, 1900. 8vo. 23 pp.

MEYER (F.). Rapporto sui progressi della teoria proiettiva degli invarianti nell'ultimo quarto di secolo. Traduzione dal tedesco di G. Vivanti con aggiunte e modificazioni dell'autore. Napoli, Pellerano, 1900. 8vo. 237 pp.

PAROMENSKY (A.). Differential and integral calculus with applications to analysis and geometry. 2d edition, revised. St. Petersburg, 1901. 8vo. 431 pp. (Russian.) M. 12.00

PINCHERLE (S.). Introduzione al corso di algebra complementare e di geometria analitica; appunti redatti per uso degli studenti. Bologna, Zanichelli, 1901. 8vo. 66 pp. Fr. 2.00

RÉPERTOIRE bibliographique des sciences mathématiques. 8e série: fiches 701 à 800. 9e série: fiches 801 à 900. Paris, Gauthier-Villars, 1900. 18mo. 200 pp. Fr. 4.00.

RUSSELL (B. A. W.). Essai sur les fondements de la géométrie. Traduction par A. Cadenat, revue et annotée par l'auteur et par L. Couturat. Paris, Gauthier-Villars, 1901. 8vo. 10 + 274 pp

SCHIFF (V.). Collection of exercises and problems in the differential and integral calculus. Parts I and II. 2d edition. St. Petersburg, 1900. 8vo. 405 + 599 pp. (Russian.) M. 15.00

SOLOMON (G.). Plane analytic geometry. 2d edition. Moscow, 1900. 8vo. 418 pp. (Russian.) M. 9.00

——. Solid analytic geometry. 2d edition. Moscow, 1900. 8vo. 297 pp. (Russian.) M. 6.50

SPORER (B.). Niedere Analysis. 2te Auflage. Leipzig, Göschen, 1901. 12mo. 176 pp. Cloth. (Sammlung Göschen, No. 53.) M. 0.80

VIVANTI (G.). See MEYER (F.).

II. ELEMENTARY MATHEMATICS.

AMANZIO (D.). Sette lezioni di algebra ad uso degli alunni d'istituto tecnico. Napoli, 1901. 8vo. Fr. 3.00

BAGNOLI (E.). Geometria rettilinea e curvilinea trattata con metodo preeuclidico e cronogoniometria. Roma, Loescher, 1901. 8vo. With atlas.

BARDEY (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik, vorzugsweise für Realschulen, Progymnasien und Realprogymnasien. In alter und neuer Ausgabe. Neue Ausgabe, nach der 10ten Auflage bearbeitet von F. Pietzker und O. Presler. Leipzig, Teubner, 1901. 8vo. 7 + 314 pp. Cloth. M. 2.60

BENDT (F.). Katechismus der Trigonometrie. 3te Auflage. Leipzig, Weber, 1901. 12mo. 8 + 135 pp. Cloth. (Weber's illustrirte Katechismen, No. 114.) M. 2.00

EGOROF (F.). See MALININ (A.).

FRANCHIS (M. DE). Elementi di geometria ad uso delle scuole tecniche. Milano, Sanchon, 1901. 12mo. 4 + 227 pp.

GHERSI (J.). Metodi facili per risolvere i problemi di geometria elementare. Milano, Hoepli, 1900. 12mo. 190 pp. Fr. 1.50

GIKA (D.) and MUROMTSEF (A.). Geometrical exercises. Part I. 6th edition. Moscow, 1900. 8vo. 188 pp. (Russian.) M. 3.00

GIKA (D.). Elements of geometry. 3d edition. Moscow, 1900. 8vo. 277 pp. (Russian.) M. 3.50

GRAHAM (J. E.). Elementary treatise on the calculus for engineering students. Second revised enlarged edition. New York, Spon & Chamberlain, 1901. 12mo. 12 + 276 pp. Cloth. $2.00

KÖNIGBAUER (J.). Geometrische Aufgaben fur Mittelschulen und Lehrerbildungsanstalten. 5te Auflage. Regensburg, Habbel, 1901. 8vo. 133 pp. Cloth. M. 1.00

KORDGIEN (H.). Das mathematische Pensum für das Einjährig-Freiwilligen-Examen. Theorie und Praxis. Aufgaben mit ausführlichen Lösungen und Erläuterungen. Teil 2: Planimetrie; Einleitung in die Stereometrie; Elemente der ebenen Trigonometrie. Berlin, Grote, 1901. 8vo. 6 + 213 pp. M. 2.70

LONGCHAMPT (A.). Recueil de problèmes tirés des compositions données à la Sorbonne, de 1853 à 1875–76, pour les baccalauréats ès sciences, suivis des compositions de mathématiques élémentaires, de physique, de chimie et de sciences naturelles données aux concours généraux de 1846 à 1875–76, et de types d'examens du baccalauréat ès lettres et des baccalauréats ès sciences. IIe partie: Géométrie. 1876–1900. 2e édition. Paris, Gauthier-Villars, 1900. 4to. Fr. 4.40

MALININ (A.) and EGOROF (F.). Geometry, with a collection of geometrical problems. 5th edition. Moscow, 1900. 8vo. 272 pp. (Russian.) M. 3.50

MICHEL (F.). Recueil de problèmes de géométrie analytique à l'usage des classes de mathématiques spéciales; solutions des problèmes donnés aux concours d'admission à l'Ecole Polytechnique, de 1860 à 1900. Paris, Gauthier-Villars, 1900. 8vo. 240 pp. Fr. 6.00

MUROMTSEF (A.). See GIKA (D.).

PIETZKER (F.). See BARDEY (E.).

PRESLER (O.). See BARDEY (E.).

SHMULEVITCH (P.). Collection of problems in trigonometry, to be used in preparing for the examinations at mining, technical, and other institutions. St. Petersburg, 1900. 8vo. 292 pp., 2 plates. (Russian.) M. 4.50

SIMASHKO (F.). Trigonometry. 5th edition. St. Petersburg, 1900. 8vo. 98 pp. (Russian.) M. 3.00

SCOTTI (G.). Elementi di aritmetica ragionata ad uso del ginnasio superiore, secondo gli ultimi programmi governativi. Torino, libreria Salesiana, 1901. 8vo. 160 pp. Fr. 1.50

SOCCI (A.) e TOLOMEI (G.). Elementi di geometria secondo il metodo di Euclide; libro di testo per i ginnasi ed i licei conforme agli ultimi programmi. Vol. I: per la IV. classe ginnasiale. 2a impressione. Firenze, Le Monnier, 1901. 8vo. 21 + 93 pp. Fr. 1.00

——. Vol. II: per la V. classe ginnasiale. Firenze, le Monnier, 1901. 8vo. Pp. 95–173. Fr. 0.75

TOBEL (E. VON). Geometrie fur Sekundarschulen. Auflösungen zu den Aufgaben. Zurich, Fussli, 1901. 8vo. 6 + 80 pp. Boards. M. 1.80

TOLOMEI (G.). See SOCCI (A.).

VOINOF (A.). Collection of problems in geometry, algebra and trigonometry. 2d edition. Moscow, 1899. 8vo. 148 pp. (Russian.) M. 3.00

WELLS (W.). Plane and spherical trigonometry. With four-place tables. Boston, Heath, 1901. 12mo. 148 pp. Half-leather. $1.08

WINTER (W.). Trigonometrie; Lehrbuch und Aufgabensammlung für Schulen. 3te Auflage. München, Ackermann, 1900. 8vo. 4 + 80 pp. Boards. M. 1.15

ZINNA (A.). L'analisi diofantea esposta con nuovi metodi ed illustrata con esempi. Trapani, Messina e Co., 1900. 16mo. 68 pp.

III. APPLIED MATHEMATICS.

BAKHUIS-ROOZEBOOM (H. W.). See LAAR (J. J. VAN).

DOMOGAROF (A.). Elements of mechanics. Part III: Statics. St. Petersburg, 1900. 8vo. 168 pp. (Russian.)

EHEMANN (G. A.). Hydraulische Motoren. Allgemeine und theoretische Darstellung der Wasserräder und Turbinen und die Theorie der Turbinen. Leipzig, Schäfer, 1901. 8vo. 76 pp., 9 plates. M. 9.00

LAAR (J. J. VAN). Lehrbuch der mathematischen Chemie. Mit einer Einleitung von H. W. Bakhuis-Roozeboom. Leipzig, Barth, 1901. 8vo. 19 + 224 pp. Cloth. M. 8.00

LA GOURNERIE (J. DE). Traité de géométrie descriptive. 3e édition. Troisième partie. Paris, Gauthier-Villars, 1901. 4to. 20 + 230 pp., 46 plates. Fr. 10.00

LANGER (H.). Ueber die mechanischen Prinzipien der modernen Elektrizitätslehre. Teil II. Linz, 1900. 8vo. 44 pp.

LEHRBUCH DER NAVIGATION, herausgegeben vom Reichs-Marine-Amt. Vol. I: Terrestrische Navigation, 354 pp. Vol. II: Astronomische Navigation, 439 pp. Vol. III: Anleitung zu Küstenvermessungen. 112 pp., 7 charts. Berlin, 1901. 8vo. M. 16.00

LÉVY (M.). Leçons sur la théorie des marées, professées au Collège de France. (En 2 volumes) Vol I: Théories élémentaires: formules pratiques de la prévision des marées. Paris, Gauthier-Villars, 1898. 4to. Fr. 14.00

MAYR (R.). Ueber Körper von kinetischer Symmetrie. (Diss.) München, 1900. 8vo. 54 pp.

OETTINGEN (A. VON). Elemente des geometrisch-perspektivischen Zeichnens. Leipzig, Engelmann, 1901. 8vo. 7 + 177 pp. Cloth. M. 9.00

SCHRÖDER (J.). Darstellende Geometrie. Teil 1. Leipzig, Göschen, 1901. 8vo. 8 + 282 pp. Cloth. (Sammlung Schubert, No. XII.) M. 5.00

STEELE (W. J.) and TAIT (P. G.). A treatise on dynamics of a particle with numerous examples. 7th edition, carefully revised. London, Macmillan, 1901. 12mo. Cloth. 12s. 6d.

TAIT (P. G.). See STEELE (W. J.).

THIEME (G.). Analytical mechanics. Part 3. St. Petersburg, 1900. 8vo. 124 pp. (Russian.) M. 4.50

WARBURG (E.). Ueber die kinetische Theorie der Gase. Berlin, 1901. 8vo. 32 pp. M. 0.80

SURFACES WHOSE FIRST AND SECOND FUNDAMENTAL FORMS ARE THE SECOND AND FIRST RESPECTIVELY OF ANOTHER SURFACE.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, February 23, 1901.)

THE fundamental theorem in the theory of surfaces is:* *Given two quadratic differential forms*

$$(1)\qquad \begin{aligned} f &= E du^2 + 2F du dv + G dv^2 \\ \varphi &= D du^2 + 2D' du dv + D'' dv^2, \end{aligned}$$

of which the first is definite, in order that there exist a surface which shall have f and φ for its first and second fundamental forms respectively, it is necessary and sufficient that the coefficients of these forms satisfy the Codazzi equations

$$(2)\qquad \begin{aligned} &\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} - \left\{\begin{matrix}12\\1\end{matrix}\right\} D + \left(\left\{\begin{matrix}11\\1\end{matrix}\right\} - \left\{\begin{matrix}12\\2\end{matrix}\right\}\right) D' + \left\{\begin{matrix}11\\2\end{matrix}\right\} D'' = 0 \\ &\frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} + \left\{\begin{matrix}22\\1\end{matrix}\right\} D + \left(\left\{\begin{matrix}22\\2\end{matrix}\right\} - \left\{\begin{matrix}12\\1\end{matrix}\right\}\right) D' - \left\{\begin{matrix}12\\2\end{matrix}\right\} D'' = 0 \end{aligned}$$

and the equation of Gauss

$$(3)\qquad \begin{aligned} &\frac{1}{\sqrt{EG - F^2}} \left[\frac{\partial}{\partial v} \left(\frac{\sqrt{EG - F^2}}{E} \left\{\begin{matrix}11\\2\end{matrix}\right\} \right) \right. \\ &\qquad \left. - \frac{\partial}{\partial u} \left(\frac{\sqrt{EG - F^2}}{E} \left\{\begin{matrix}12\\2\end{matrix}\right\} \right) \right] = \frac{DD'' - D'^2}{EG - F^2}; \end{aligned}$$

where the Christoffel symbols $\left\{\begin{matrix}rs\\t\end{matrix}\right\}$ are formed with respect to the form f. And, moreover, *when these conditions are satisfied the corresponding surface is unique and determinate.*

We propose now to determine those surfaces S with fundamental forms (1) which have associated with them surfaces S_1, whose first and second fundamental forms are, respectively,

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* Bianchi, Lezioni, p. 92.

$$Ddu^2 + 2D'dudv + D''dv^2,$$
$$Edu^2 + 2Fdudv + Gdv^2.$$

Denoting by subscript $_1$ functions belonging to S_1, we have

$$(4)\qquad \begin{aligned} f_1 &= ds_1^2 = E_1du^2 + 2F_1dudv + G_1dv^2, \\ \varphi_1 &= D_1du^2 + 2D_1'dudv + D_1''dv^2. \end{aligned}$$

Comparing these with the above, we remark that for a surface S_1 satisfying the conditions of the problem,

$$(5)\quad E_1 = D,\ F_1 = D',\ G_1 = D'';\ D_1 = E,\ D_1' = F,\ D_1'' = G.$$

It is evident that the double system of lines which is conjugate for S and S_1 is composed of the lines of curvature on each surface. We will refer both surfaces to these lines; then

$$(6)\qquad F = 0, \qquad D' = 0.$$

With this special choice of parametric lines the Codazzi equations become

$$(7)\qquad \begin{aligned} &\frac{\partial D}{\partial v} - \tfrac{1}{2}\frac{\partial E}{\partial v}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0, \\ &\frac{\partial D''}{\partial u} - \tfrac{1}{2}\frac{\partial G}{\partial u}\left(\frac{D}{E} + \frac{D''}{G}\right) = 0; \end{aligned}$$

and the Gauss equation

$$(8)\quad \frac{DD''}{\sqrt{EG}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = 0.$$

In order that S_1 may be a surface the coefficients of its fundamental forms must satisfy these equations. Hence in order that S may be the kind of surface sought, it is necessary that its fundamental coefficients satisfy the conditions

$$(9)\qquad \begin{aligned} &\frac{\partial E}{\partial v} - \tfrac{1}{2}\frac{\partial D}{\partial v}\left(\frac{E}{D} + \frac{G}{D''}\right) = 0, \\ &\frac{\partial G}{\partial u} - \tfrac{1}{2}\frac{\partial D''}{\partial u}\left(\frac{E}{D} + \frac{G}{D''}\right) = 0, \end{aligned}$$

and

$$(10)\quad \frac{EG}{\sqrt{DD''}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{D}}\frac{\partial\sqrt{D''}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{D''}}\frac{\partial\sqrt{D}}{\partial v}\right) = 0.$$

Substituting in equations (7) the expressions for $\frac{\partial E}{\partial v}$ and $\frac{\partial G}{\partial u}$, as given by equations (9), we get in each case

$$\left(\frac{D}{E}+\frac{D''}{G}\right)\left(\frac{E}{D}+\frac{G}{D''}\right)=4. \tag{11}$$

Denote by ρ_1 and ρ_2 the principal radii of curvature at the point (u, v) on S; then

$$\frac{1}{\rho_1}=\frac{D}{E}, \quad \frac{1}{\rho_2}=\frac{D''}{G}.$$

When these expressions for ρ_1 and ρ_2 are replaced by the latter in the equation of condition (11), it becomes

$$\frac{1}{\rho_1\rho_2}=\tfrac{1}{2}\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}\right). \tag{12}$$

Hence S must be a surface for which the total curvature at a point and the Casorati * curvature are equal. Developing equation (12) we remark that it is necessary that

$$\rho_1=\rho_2.$$

Hence, in order that the first and second fundamental forms of a surface S may determine a surface for which these forms shall be respectively the second and first forms it is necessary that S have its two radii equal and of the same sign.

This class of surfaces has been the object of a study by Monge,† and later by Forsyth.‡ The former has shown that the sphere is the only *real* surface belonging to the class. His discussion was with reference to the differential equation of the second order, characteristic of such surfaces, viz.

$$\begin{aligned}&[(1+q^2)r-2pqs+(1+p^2)t]^2\\&\quad=4(rt-s^2)(1+p^2+q^2),\end{aligned} \tag{13}$$

for the determination of which the surface was given by the equation

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* "Mésure de la courbure des surfaces suivant l'idée commune", *Acta Mathematica*, 14, 2.

† Application de l'analyse à la géométrie, 5th ed., pp. 196–211.

‡ "Note on surfaces whose radii of curvature are equal and of same sign," *Messenger of Mathematics*, new series, no. 321.

$$(14)\qquad z=f(x,y),$$

and where, as originally denoted by Monge,

$$(15)\qquad p,\ q,\ r,\ s,\ t=\frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y},\ \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial x\partial y},\ \frac{\partial^2 z}{\partial y^2}.$$

Forsyth integrated the same equation (13) by a process differing from that used by Monge and found that all such surfaces are given by eliminating μ between the equations

$$16)\qquad y+\mu x=\varphi(\mu),\ z-ix(1+\mu^2)^{\frac{1}{2}}=\psi(\mu)$$

where φ and ψ are arbitrary functions. We shall now determine the forms of the functions, which correspond to surfaces furnishing a solution to our problem.

To this end we remark that the notations of Gauss and of Monge have the following relations:

$$E=1+p^2,\qquad F=pq,\qquad G=1+q^2,$$

$$(17)\qquad D=\frac{r}{\sqrt{1+p^2+q^2}},$$

$$D'=\frac{s}{\sqrt{1+p^2+q^2}},\quad D''=\frac{t}{\sqrt{1+p^2+q^2}}.$$

It is evident then that our problem reduces to the finding of φ and ψ such that

$$(18)\qquad \frac{r,\ s,\ t}{\sqrt{1+p^2+q^2}}=1+p_1^2,\ p_1q_1,\ 1+q_1^2,$$

and

$$(19)\qquad 1+p^2,\ pq,\ 1+q^2=\frac{r_1,\ s_1,\ t_1}{\sqrt{1+p_1^2+q_1^2}},$$

where the subscript $_1$ denotes that the functions belong to the associated surface S_1.

Substituting for the functions in (18) their expressions as derived from equations (16) and from the similar equations

$$y+\mu_1 x=\varphi_1(\mu_1),$$
$$z-ix(1+\mu_1^2)^{\frac{1}{2}}=\psi_1(\mu_1),$$

corresponding to the surface S_1, we find on equating to zero the coefficients of x^2, x, and 1, the following relations:

$$(20)\quad \begin{cases} \dfrac{-i\mu^2 A}{(1+\mu^2)^{\frac{3}{2}}} = \dfrac{\mu_1^{\,2}}{1+\mu_1^{\,2}}, \\ -\dfrac{2\mu}{(1+\mu^2)^{\frac{1}{2}}} - \mu^2 AB = 2\Bigg[\varphi_1'\mu_1^{\,2} \\ \quad + i\mu_1(i\varphi_1'\mu_1 - \sqrt{1+\mu_1^{\,2}}\psi_1') + \dfrac{i\mu_1^{\,3}\psi_1'}{\sqrt{1+\mu_1^{\,2}}}\Bigg], \\ \dfrac{2\varphi'\mu}{\sqrt{1+\mu^2}} + A\mu^2(\varphi'\psi'' - \varphi''\psi') \\ \quad = -\varphi_1'^{\,2}\mu_1^{\,2} + 2i\mu_1\sqrt{1+\mu_1^{\,2}}\varphi_1'\psi_1' + \mu_1^{\,2}\psi_1'^{\,2}; \end{cases}$$

$$(21)\quad \begin{cases} \dfrac{-i\mu A}{(1+\mu^2)^{\frac{3}{2}}} = \dfrac{\mu_1}{1+\mu_1^{\,2}}, \\ -\dfrac{1}{\sqrt{1+\mu^2}} - \mu AB \\ \quad = i(i\varphi_1'\mu_1 - \sqrt{1+\mu_1^{\,2}}\psi_1') + \dfrac{2i\mu_1^{\,2}\psi_1'}{\sqrt{1+\mu_1^{\,2}}}, \\ \dfrac{\varphi'}{\sqrt{1+\mu^2}} + A\mu(\varphi'\psi'' - \varphi''\psi') \\ \quad = i\sqrt{1+\mu_1^{\,2}}\varphi_1'\psi_1' + \mu_1\psi_1'^{\,2}; \end{cases}$$

$$(22)\quad \begin{cases} \dfrac{-iA}{(1+\mu^2)^{\frac{3}{2}}} = \dfrac{1}{1+\mu_1^{\,2}}, \\ -AB = -2\varphi_1' + \dfrac{2i\mu_1}{\sqrt{1+\mu_1^{\,2}}}, \\ A(\varphi'\psi'' - \psi'\varphi'') = \varphi_1'^{\,2} + \psi_1'^{\,2}, \end{cases}$$

where, for the sake of brevity, we have written

$$(23)\quad \begin{aligned} A &= \frac{1}{i\mu\varphi' + \sqrt{1+\mu^2}\psi'}, \\ B &= \psi'' - \varphi' i(1+\mu^2)^{-\frac{3}{2}} + i\mu(1+\mu^2)^{-\frac{1}{2}}\varphi''. \end{aligned}$$

The similar relations arising from equations (19) can be gotten at once from the above by giving subscripts to the terms not having them and removing them from those which have; we shall refer to these relations as (20′), (21′) and (22′), respectively.

From the first relations of (20), (21), and (22) we see that

$$\mu_1 = \mu,$$

hence these three relations and the corresponding ones in (20′), (21′) (22′) may be replaced, respectively, by

$$(24)\qquad \begin{aligned} \mu\varphi' - i\sqrt{1+\mu^2}\psi' &= -\frac{1}{\sqrt{1+\mu^2}}, \\ \mu\varphi_1' - i\sqrt{1+\mu^2}\psi_1' &= -\frac{1}{\sqrt{1+\mu^2}}. \end{aligned}$$

Combining these with the second relations of (20), (21), (22) and (20′) (21′), (22′), we find that the latter may be replaced by

$$(25)\qquad \begin{aligned} i\sqrt{1+\mu^2}[\psi'' - i(1+\mu^2)^{-\frac{3}{2}}\varphi' + i\mu\,(1+\mu^2)^{-\frac{1}{2}}\varphi''] \\ = +\,2\varphi_1' - \frac{2i\mu}{\sqrt{1+\mu^2}}, \\ i\sqrt{1+\mu^2}[\psi_1'' - i(1+\mu^2)^{-\frac{1}{2}}\varphi_1' + i\mu(1+\mu^2)^{-\frac{3}{2}}\varphi_1''] \\ = +\,2\varphi' - 2i\mu(1+\mu^2)^{-\frac{1}{2}}. \end{aligned}$$

Differentiating the second of equations (24) and combining the result with the second of equations (25) and both of (24), we find

$$(26)\qquad \varphi_1' = \varphi', \qquad \psi_1' = \psi',$$

hence φ_1 and ψ_1 can differ at most from φ and ψ, respectively, by an additive constant. Hence *S and S_1 are the same surface, to a translation près.*

Making use of the preceding results, we find that the third equations of (20), (21), (22), (20′), (21′), and (22′) are equivalent to the single one

$$(27)\qquad i\sqrt{1+\mu^2}(\varphi'\psi'' - \psi'\varphi'') = \varphi'^2 + \psi'^1.$$

Expressing this relation in terms of φ and its derivatives, we find

$$(28)\qquad \varphi'' = \frac{1}{(1+\mu^2)^{\frac{3}{2}}},$$

from which we get

$$(29)\qquad \varphi = \sqrt{1+\mu^2} + C_1\mu + C_2,$$

where C_1 and C_2 are arbitrary constants. On substituting this expression in (24), we find

$$i\psi = C_1\sqrt{1+\mu^2} + \mu + C_3, \tag{30}$$

C_3 being a third arbitrary constant. From (29) and (30) we get

$$\begin{aligned} \Phi &= \varphi + i\psi = (C_1 + 1)(\mu + \sqrt{1+\mu^2}) + C_2 + C_3, \\ \Psi &= \varphi - i\psi = (C_1 - 1)(\mu - \sqrt{1+\mu^2}) + C_2 - C_3. \end{aligned} \tag{31}$$

It is readily seen that equations (16) can be replaced by

$$\begin{aligned} y + iz + (\mu + \sqrt{1+\mu^2})x &= \Phi, \\ y - iz + (\mu - \sqrt{1+\mu^2})x &= \Psi, \end{aligned} \tag{32}$$

where Φ and Ψ have the expressions (31). Writing in these expressions

$$C_1 = a, \qquad C_2 = b, \qquad C_3 = ic,$$

and eliminating μ from the equations (32) thus obtained we get

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = 1;$$

that is, a sphere of center (a, b, c) and radius unity. Moreover, only when the radius is unity are the conditions satisfied.

We have then the following results:

The ruled surfaces, defined by the equations

$$\begin{aligned} y + \mu x &= \sqrt{1+\mu^2} + C_1\mu + C_1 \\ z - ix\sqrt{1+\mu^2} &= \mu + C_1\sqrt{1+\mu^2} + C_3, \end{aligned}$$

are the only surfaces whose first and second fundamental forms can be taken for the second and first fundamental forms of a surface. Further the second surface is only the first to a translation près. And of these surfaces the only real one is the sphere of radius unity.

PRINCETON UNIVERSITY,
February 14, 1901.

ON THE GROUPS GENERATED BY TWO OPERATORS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, April 27, 1901.)

It is well known that the only abelian groups which are generated by two operators are those which have just two independent generators.* If the order of such a group is p^m (p being any prime number), it must be of type $(m-\alpha, \alpha)$. In general, if the order of any one of these abelian groups is written in the form $2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ ($p_1, p_2, \cdots$ being odd prime numbers) it is clear that at least one of the subgroups of orders $2^{\alpha_0}, p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$ contains just two independent generators while the remaining ones (if any) are cyclic.

When the two generating operators are not commutative the matter becomes much more difficult. A very interesting and simple case presents itself when each of these operators (s_1, s_2) is of order two. From the equation $s_1 s_2 = s_3$ we obtain the following:

$$s_2 = s_1 s_3 \qquad s_2 s_1 = s_1 s_3 s_1 = (s_1 s_2)^{-1} = s_3^{-1} = s_2 s_3 s_2.$$

Since s_1 and s_2 transform s_3 into its inverse they generate the group of dihedral rotations.† By properly selecting s_1 and s_2 the order of s_3 can be made any positive integer whatever.‡ That is, *every group that is generated by two operators of order two is a dihedral rotation group and every dihedral rotation group is generated by two operators of order two.* When the order of s_3 is two this group is abelian and vice versa.

From the known theorems which we proceed to give, it follows almost directly that every symmetric group and every alternating group is generated by two operators. (1) If a primitive group whose degree exceeds 8 contains a substitution whose degree is less than 6, it is alternating or symmetric.§ (2) If a transitive group of degree n contains a substitution whose order is a prime number $p > n/a$ (a being the smallest factor of n) it is primitive. (3) If two substitutions have only one common element, their commutator is of degree three, and if they have two consecutive elements in common, without being commutative, their commutator is of degree four and order two.* (4) There is at least one prime number between m and $2m - 2$, whenever $m > 7/2$.†

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* Frobenius and Stickelberger, *Crelle*, vol. 86, 1879, p. 217. Only non-cyclic groups are considered in the present note.

† Cf. Klein, Ikosaeder, 1884, p. 9.

‡ American Journal of Mathematics, vol. 22, 1900, p. 185.

§ Netto-Cole, Theory of substitutions, 1892, p. 138; BULLETIN, vol. 4, 1898, p. 141.

As a direct consequence of the last theorem we have the theorem : The alternating group G of degree n contains a substitution S_1 of prime order $p > n/2$. From the other three theorems it follows that S_1 and any circular substitution S_2 which involves all the elements of G that are not contained in S_1 together with either one or two elements of S_1, generate the symmetric group when S_2 is negative and the alternating group when S_2 is positive. If S_2 is positive when it has one element in common with S_1, it must be negative when it has two elements in common. Hence every symmetric and every alternating group whose degree exceeds 8 is generated by two operators. It is easy to verify that this theorem is also true when $n \leqq 8$‡.

It has been observed that two operators of order two generate a group which is well known. The next case in the order of simplicity presents itself when one of the generating operators S_1 is of order three while the other S_2 is of order two. When their product is of order two, three, four, or five, these operators generate the symmetric group of order 6, the alternating group of order 12, the symmetric group of order 24, or the alternating group of 60 respectively.§ We proceed to prove that S_1 and S_2 may be so chosen that they generate the alternating group and also so that they generate the symmetric group of every degree $n > 8$. Let

$$s_1 = a_1a_2a_3 \cdot a_4a_5a_6 \cdot \cdots \cdot a_{\beta-2}a_{\beta-1}a_\beta, \quad s_2' = a_3a_4 \cdot a_6a_7 \cdot \cdots \cdot a_{\beta-3}a_{\beta-2}.$$

It is clear that s_1s_2' is a circular substitution of degree β. We shall form s_2 by adding suitable transpositions to s_2'. In what follows $p = n - \alpha$ will be used to represent a prime number such that $n - 2 > p > \frac{n}{2}$ and it will be assumed that $n > 11$ unless the contrary is stated.

When $n = \beta$ and α is odd, we may obtain generators of the alternating and the symmetric group by multiplying s_2' by $a_1a_{\frac{3\alpha+1}{2}}$ and $a_1a_{\frac{3\alpha-5}{2}} \cdot a_{n-4}a_{n-1}$ respectively ; when α is even, the corresponding multipliers are $a_1 a_{\frac{3\alpha-2}{2}} \cdot a_{\beta-1} a_\beta$ and $a_2 a_{\frac{3\alpha+4}{2}}$ respectively. By using the s_2 thus obtained and the given s_1, we can therefore generate any alternating or symmetric group whose degree exceeds 9 and is divisible by 3. When $n = \beta + 1$ and α is odd, we may multiply s_2' by $a_1 a_{\frac{3\alpha+1}{2}} \cdot a_\beta a_{\beta+1}$ and $a_1 a_{\frac{3\alpha-5}{2}} \cdot a_\beta a_{\beta+1} \cdot a_{\beta-4} a_{\beta-1}$ respectively to obtain the required s_2; while the corresponding multipliers are $a_1 a_{\frac{3\alpha-2}{2}} \cdot a_{\beta-4} a_{\beta+1} \cdot a_{\beta-1} a_\beta$ and $a_1 a_{\beta+1} \cdot a_2 a_{\frac{3\alpha+4}{2}}$ when α is even.* Finally, when $n = \beta + 2$ and α is odd, suitable factors of s_2' are $a_1 a_{\frac{3\alpha+1}{2}} \cdot a_{\beta-1} a_{\beta+1} \cdot a_\beta a_{\beta+2}$ and $a_1 a_{\frac{3\alpha-5}{2}} \cdot a_{\beta-7} a_{\beta-4} \cdot a_{\beta-1} a_{\beta+1} \cdot a_\beta a_{\beta+2}$; when α is even the factors are $a_1 a_{\frac{3\alpha-2}{2}} \cdot a_{\beta-7} a_{\beta+2} \cdot a_{\beta-4} a_{\beta+1} \cdot a_{\beta-1} a_\beta$ and $a_1 a_{\beta+1} \cdot a_2 a_{\frac{3\alpha+4}{2}} \cdot a_\beta a_{\beta+2}$, respectively.

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* Bochert, *Math. Annalen*, vol 40, 1892, p. 176.

† Tchebichef, *Pétersb. Mem.*, vol. 6 (1851); *Liouville*, vol. 17 (1852).

‡ Cf. Netto, *loc. cit.*, p. 90.

§ Cf. Burnside, Theory of Groups, 1897, p. 291.

When $n < 12$, it is easy to examine the cases directly and thus prove that *every alternating group with the exception of those of degrees* 3, 6, 7, 8 *is generated by two of its substitutions of orders two and three respectively, and every symmetric group with the exception of those of degrees* 5, 6, 8 *is generated by two of its substitutions of the same orders.* Hence 6 and 8 are the only degrees for which neither the symmetric nor the alternating group is generated by two of its substitutions of orders two and three respectively. The only transitive groups generated in these two cases are of orders 6, 12, 18, 24, 48, 60, 168 and 336.

A CURIOUS APPROXIMATE CONSTRUCTION FOR π.

BY MR. GEORGE PEIRCE.

(Read before the American Mathematical Society, April 27, 1901.)

CONSTRUCTION: AC, BD are perpendicular diameters of a circle. $DE = AO$. CE produced cuts BD in G and the circle in F.

To prove that $\pi r = FC + BG$ approximately.

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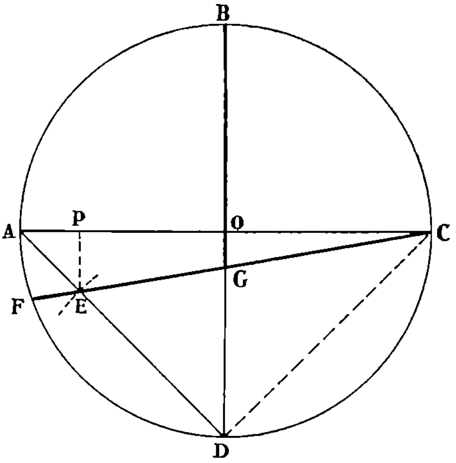
* When $n = 13$ we may assume $\alpha = 2$, *Quar. Jour. of Math.*, vol. 29 (1898), p. 228.



Draw EP perpendicular to AC. Then $CE = r\sqrt{3}$. Also

$$FE = AE \cdot ED/EC = r(\sqrt{2}-1)/\sqrt{3}.$$

Therefore

$$FC = CE + FE = \tfrac{1}{3}r(\sqrt{6}+2\sqrt{3}).$$

Again

$$OG = PE \cdot OC/PC = PE \cdot OC/\sqrt{EC^2 - PE^2}$$

$$= r(1-\tfrac{1}{2}\sqrt{2})/\sqrt{3-(1-\tfrac{1}{2}\sqrt{2})^2} = r(3-2\sqrt{2}),$$

and

$$BG = BO + OG = r(4-2\sqrt{2})$$

Hence

$$FC + BG = r(4-2\sqrt{2}+\tfrac{2}{3}\sqrt{3}+\tfrac{1}{3}\sqrt{6}) = 3.14277r.$$

Now $\pi = 3.14159$. Hence the error is less than 1/2600 of the whole; in a circle of 75 yards radius the error would be about one inch.

HAVERFORD COLLEGE.

NON-EUCLIDEAN GEOMETRY.

Non-Euclidean Geometry. By HENRY PARKER MANNING, Ph.D. Boston, Ginn & Co., 1901. 16mo., pp. 95.

THIS work is, as far as we know, the first original book upon this subject that has been published in English. Perhaps, the adjective "original" is misleading, for the author, in his preface, disclaims any attempt at originality. His object is to collect a number of the simpler theorems of non-euclidean geometry, and present them in compact and logical form to readers of slight mathematical knowledge.

At the outset, he avoids a dangerous pit-fall: metaphysics. To pass over in silence all of those philosophical questions which lie at the base of geometry, would be unpardonable in a book of more ambitious nature, but in this case, the omission seems wise. A beginner can extract but little profit from discussions of the foundations of geometry. He will accept with equal gladness Russell's contention that the conception of geometrical equality depends upon that of rigid motion, or Veronese's view that rigid motion presupposes a continuous succession of geometrically equal figures. At the same time our author makes a mistake in throwing upon the elementary text books the responsibility for most of the fundamental definitions and assumptions. We are left in doubt, for instance, whether he considers a line as the path of a moving particle or the boundary of a surface. It seems also a mistake to lay down, as universally valid, the axiom that two straight lines can meet but once, for spherical geometry is thus excluded. It is safe to say that spherical geometry is of quite as much importance as elliptic, and the study of such a figure as the sphere with two centers, might well prove attractive to beginners.* A wiser plan would be to put this axiom among the assumptions for restricted figures, and then prove that if space may be moved as a whole, two straight lines can not intersect more than twice.†

The general arrangement of the book seems to us excellent. The author first brings out the points of similarity of his three kinds of geometry, then develops their individual peculiarities. Moreover, the analytic work is placed at the end, out of consideration for those readers who prefer to avoid trigonometry, and are stampeded by the calculus.

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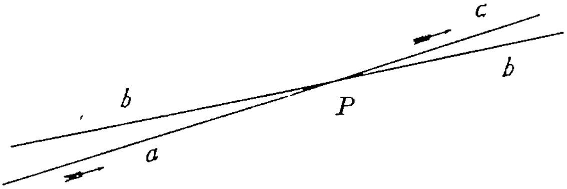
* Conf. Veronese, Grundzüge der Geometrie von mehreren Dimensionen (German translation by Schepp), p. 503.

† Killing, Grundlagen der Geometrie, p. 56.

The first chapter deals with "Pure Geometry" and gives a number of propositions for both unlimited and restricted figures, which are universally valid under the axioms laid down. Then the three hypotheses concerning the fourth angle of a trirectangular quadrilateral are stated and carried along side by side to the end of the chapter.

The second chapter deals with "Hyperbolic Geometry" and develops the theory of intersecting, parallel, and non-intersecting lines in a most elementary way, with a short notice of boundary and equidistant curves.

The third chapter, which consists of five pages only, is devoted to "Elliptic Geometry." Here it must be noticed that the author passes over in absolute silence the most striking peculiarity of the elliptic plane; namely that it is a unilateral or double surface. Now not only is this a matter of great importance in itself, but there lurks here a very serious difficulty for the beginner, which may be stated as follows: Suppose that in the elliptic plane we have two straight lines a and b lying close to one another throughout their whole extent.



Let a man start from their intersection P and make a complete circuit of the line a. If he starts in the right direction he will notice that b will at first lie on his right hand; yet when he has nearly completed the circuit and is approaching P again, b will seem to be on his left, and the change will have occurred without the two lines crossing in the meanwhile. Of course the explanation is that when he draws near to P at the end of the journey his head is pointing in the opposite direction from what it was before; but it is scarcely fair to assume that a beginner will be sharp enough to think this out.

The fourth chapter gives the analytic development of what has gone before and the book closes with a very short historical note.

The merits of the book are, then, a judicious choice of subject matter, and logical arrangement. Also the author has the rare gift of being interesting. The defects consist in carelessness in matters of detail, and occasional lack of rigor in demonstration. For instance, the whole analytic work is based on the connection between plane and spherical triangles. In Chapter I, part 3, theorem 11, we find the statement that spherical trigonometry is the same under all three hypotheses, and the reason given is that the relations of spherical figures depend merely upon the magnitude of plane and diedral angles. But the connection between a spherical and a diedral angle comes through the plane angle of the latter; what right have we to assume that two plane angles of the same diedral angle are equal? The proof given in all of the elementary books with which we are familiar is based on parallel lines, and hence is inadmissible in the present case. Incidentally, all propositions concerning mutually perpendicular planes stand or fall with the equality of these plane angles.

Fortunately, it is true that in non-euclidean geometry two plane angles of the same diedral angle are equal. An easy proof may be found from the projective definition of the magnitude of an angle with regard to the Absolute. Or a proof more suited to the present work might be developed on the following lines: The magnitude of a diedral angle may be shown to vary directly with the plane angle at any point. Hence two plane angles of the same diedral angle have a constant ratio. When one plane angle is 0, the other is an arbitrary multiple of 2π which we are at liberty to assume is 0 also. Then when one angle is π, the other is π also. Hence the factor of proportionality is 1, or the two angles are always equal.

The treatment of continuity is obscure. Axiom 4 states that "A geometrical magnitude, for example an angle, or the length of a portion of a line, varying from one value to another, passes through all intermediate values." This seems to us almost romantically vague. Moreover, unless we embody the axiom itself in our definition of the word "vary," it is not true. For instance, if we mean by an angle of a triangle an internal angle, as is habitual in elementary books, and if the base of a triangle is fixed, while the opposite vertex describes a circumference upon the base as a chord, the angle will vary from a given value to its supplement, without passing through any intermediate value. Another example of carelessness in treating continuity occurs on page 13 where we find the statement that if a line move so as always to cut off equal distance from the feet of two perpendiculars to a given line, the angles which it makes with these perpendiculars vary continuously. This is of fundamental importance, as hereon rests the proof that the same quadrilateral-hypothesis must hold everywhere in the same plane, but it should not be left without proof, merely because it is plausible, or because we ardently wish it to be true.

In the historical note at the end are serious errors. The author states (p. 92) that Saccheri, after developing elliptic, hyperbolic, and parabolic geometries side by side, states dogmatically that the two first are false. Now this is not only incorrect, but it conveys an unfair impression of the Jesuit mathematician. He does not state dogmatically that these geometries are false, but gives an elaborate and ingenious sequence of propositions in each case, to show that the system is self-contradictory.* Again the author states on the following page, that elliptic geometry was discovered by Riemann. This seems to us rash. It is by no means clear whether Riemann looked upon the geodesics of a surface of constant positive curvature as cutting in one or two points: the probability being that he held the latter view, so that it is customary to credit Klein and Newcomb with the discovery of elliptic geometry.†

We may say, in conclusion, that the book is a contribution to educational rather than to mathematical literature. It is neither a scholarly nor a profound work, but comes in answer to a real need, and marks a step in advance.

JULIAN LOWELL COOLIDGE.
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BIANCHI'S DIFFERENTIAL GEOMETRY.

Vorlesungen über Differentialgeometrie. Von LUIGI BIANCHI. Autorisierte deutsche Uebersetzung von MAX LUKAT. Leipzig, B. G. Teubner, 1896–1899. Pp. xvi + 659.

IN 1886 Bianchi published a lithographed edition of his lectures on differential geometry given at the University of Pisa during the winter 1885–86. This publication, on which the book now before us is based, consisted of only fourteen chapters. The *Vorlesungen* contains twenty-two. Professor Loria, of Genoa, says of the lithographed edition :‡

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* Conf. Engel und Staeckel, Die Theorie der Parallellinien von Euklid bis auf Gauss, pp. 67 and 109.

† Killing, loc. cit., p. 70. Russel, The Foundations of Geometry, pp. 39, 40.

‡ *Jahrbuch über die Fortschritte der Mathematik*, 1886, p. 648.

"In our opinion it fills a lamentable gap in mathematical literature * * * We know no book which treated *ex professo* infinitesimal geometry. * * * In these noteworthy lectures we have to admire not only the clearness of presentation, the elegance of the calculations, and the great number of interesting examples, but also the selection and the coördination of the material." It will perhaps be of interest to give the headings of the fourteen chapters of this first edition. I., Space curves; II., Curvilinear coördinates on surfaces, conformal representation; III., Geodesic lines, geodesic curvature; IV., Curvature of surfaces; V., Surfaces defined by their linear element; VI., Developable surfaces; VII., Surfaces one of whose radii of curvature is a function of the other; VIII., Surfaces of constant curvature; IX., Surfaces whose mean curvature is constant; X., Surfaces defined by properties of their lines of curvature; XI., Ruled surfaces; XII., Curvilinear coördinates in space, triply orthogonal systems of surfaces; XIII., Elliptic coördinates, geodesic lines on the ellipsoid; XIV., Triply orthogonal Weingarten systems; Appendix, Codazzi's formulas.

In 1894 appeared the enlarged and improved Geometria differenziale, published by Spoerri in Pisa, containing twenty chapters. The mathematical situation was then different, for during these eight years had appeared the first three volumes of Darboux's Leçons sur la théorie générale des surfaces,* a work which in the words of Bianchi "gives a complete presentation of all the results obtained in the field of infinitesimal geometry." Bianchi's book has none the less its place, and that an honorable one. His aim as stated in his preface is entirely different from Darboux's. Bianchi's purpose is "to confine himself to an explanation of the principles of the theory, and to its chief applications; to collect in a volume of moderate compass everything necessary to enable beginners * * * themselves to read the original memoirs." That Bianchi has succeeded in this task is evident. That this purpose is indeed different from Darboux's will appear to all who read the latter's book.

The Geometria differenziale is not merely the lithographed edition with six additional chapters. Apparently the whole book has been re-written, and the subjects are taken up in somewhat different order and in different connections. I cannot compare this book in any detail with the original lithographed edition, for I have not been able to obtain a copy of the latter.

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* Referred to in this review as "Darboux."

Shortly after the appearance of the Geometria differenziale, a German translation this work was undertaken by Max Lukat. The translation was published under the title of Vorlesungen über Differentialgeometrie. This edition appeared in three parts, of which the first was published in 1896, the last in 1899. All references in the following pages are to the German edition.

I shall now consider in some detail the contents of the Geometria differenziale. The first chapter on "Curves of double curvature" is a brief and complete exposition of the principal properties of space curves. In it are discussed much the same questions that are taken up by Picard in Chapter XI. and in the following chapters of the first volume of his Traité d'analyse. Bianchi's treatment is usually concise and clear. It is noticeable, however, that he uses for the direction cosines of the principal directions of a curve at any point the expressions $\cos \alpha$, $\cos \beta$, etc., instead of employing, as is customary with French writers, single letters for these quantities. It seems to me that the French practice is to be preferred.

In this chapter there is no explanation of the term "edge of regression," in the treatment of developable surfaces. Such an explanation as is to be found for example in Picard, l. c., I., p. 298, would add to the value of the discussion. It is perhaps worth mentioning that Bianchi does not suppose his coördinate axes disposed as French and German writers generally do, but adopts the so-called "English disposition."

In the second chapter we have the analytic foundation of the book, a study of quadratic differential forms. Bianchi defines the first, second, and mixed differential parameters, and states in a remark (p. 47) that all the differential parameters of two arbitrary functions U and V, formed from a binary quadratic differential form, may be obtained by repeated applications of the symbols Δ_1, Δ_2, and ∇, which indicate the first, second, and mixed differential parameters, respectively. For a proof he refers to Darboux, III., p. 260. The reference should be to page 203. This mistake in the Italian is not corrected in the translation. The proof given by Darboux is simple and short, and might with advantage have been reproduced by Bianchi. The several symbols of Christoffel are then defined. These symbols are used throughout the book. As the author says in his preface, they have the advantage of giving the formulas a simple and elegant appearance which makes them easier to remember. While I agree fully with this statement, I am not yet convinced that the use of these symbols is entirely desirable. To one not accustomed to them, they have always a strange look, and surely many an American reader, at least, must turn back to the definitions if ever he wants to know what a formula really means. It is questionable too if it is desirable to memorize the complicated formulas of differential geometry. On the other hand it may be urged that the formulas would be of great complexity and of ungainly appearance without the use of Christoffel's symbols. Often that is the case. It would be easy to mention books where it is so. But Darboux does not use them, and I think no book is more elegant in style and appearance than his. It is finally a matter of taste. The chapter contains also a study of the equivalence of two quadratic forms. One other criticism of this very important chapter must be made. The whole chapter would be much more readily understood by the beginner, and so more in accord with the purpose of the book, if the theory had been developed for binary forms instead of for forms in n variables. The binary form is the only one used in the study of surfaces. Let those who will study surfaces in n dimensions generalize for themselves, and let the student of surfaces in space not be forced to swallow what he cannot digest.

The next seven chapters (III.–IX.) contain an admirable development of the general theory of surfaces. Their headings are as follows: III., Curvilinear coördinates on surfaces, conformal representation; IV., The fundamental equations of the theory of surfaces; V., Spherical representation after Gauss, plane coördinates; VI., Geodesic curvature, geodesic lines; VII., Applicable surfaces; VIII., Deformation of ruled surfaces; IX., Evolute surfaces and the theorem of Weingarten.

Bianchi's consistent point of view in these chapters, and indeed throughout the book, is that the study of surfaces is the study of two simultaneous quadratic differential forms; the "first fundamental form" being the square of the linear element

$$ds^2 = dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2;$$

the "second fundamental form" is

$$\begin{aligned} Xd^2x + Yd^2y + Zd^2z &= -(dXdx + dYdy + dZdz) \\ &= Ddu^2 + 2D'dudv + D''dv^2. \end{aligned}$$

The letters X, Y, Z denote the direction cosines of the normal to the surface at the point (x, y, z). These seven chapters are admirably done, and contain thoroughly clear discussions of many most interesting and fundamental problems. I feel that the author is to be taken to task, however, for advising, in his preface, the beginner to omit the eighth chapter (on the deformation of ruled surfaces). It is not an especially difficult chapter, but it is an especially interesting one, and no one should omit it. Chapter III. deals with the fundamental properties of curvilinear coördinates on a surface. The conformal representation of a surface upon a plane or upon another surface is then studied by means of isothermal systems of coördinates. In the study of isothermal systems, the author omits to prove that it is necessary, as well as sufficient, in order that the curves (u, v) form an isothermal system, that the linear element in terms of these coördinates be

$$ds^2 = \lambda(Udu^2 + Vdv^2).$$

A simple proof of the necessity of this condition is given by Darboux, Volume I., p. 146. Bianchi makes no use of "symmetric coördinates," which are the parameters of the minimal curves of the surface. These coördinates seem to me to furnish the simplest method of treating the problems of conformal representation. Bianchi fails also to refer to the subject of the ordinary maps of a sphere, such as Mercator's projection. This subject furnishes one of the most direct and interesting applications of conformal representation. The chapter closes with a study of stereographic projection, where the author shows, following Cayley, that the movements of a sphere on itself may be represented by a linear substitution with constant coefficients.

The fourth chapter begins the discussion of the two fundamental forms of the surface. The formulas connecting the coefficients E, F, G, D, D', D'' of the two forms, due to Mainardi and Codazzi, are deduced. There follows a clear and simple discussion of the problem of determining the surface, having given the coefficients of the fundamental forms. This discussion seems to me one of the best things in the book. One footnote in this chapter will hardly please the reader. On page 95 we are referred for a proof of a simple theorem in differential equations, "for example, [to] C. Jordan, Cour d'analyse, vol. III." Such an indefinite reference is of no value, and is even irritating. Moreover, I cannot find that Jordan gives explicitly the theorem referred to.

The last part of the chapter contains applications of the results of the first part to proofs of the theorems of Euler and Meusnier on the curvature of the curves of a surface passing through any point of the surface. The celebrated theorem of Gauss on the total curvature of a surface at any point is then proved. Finally, the indicatrix of Dupin is discussed. Bianchi departs from what is, I believe, the general usage, in calling the indicatrix at a hyperbolic point of a surface the *two* conjugate hyperbolas. By so doing there is a certain loss of generality in the definition of the indicatrix. In the *Vorlesungen* are inserted in the proper place (p. 114) the differential equations of the lines of curvature and of the asymptotic lines of a surface having its equation in the form, $z = f(xy)$. In the Italian these equations were accidentally omitted, and were printed at the end of the book.

Chapter V. treats Gauss's spherical representation of the surface, and tangential or plane coördinates. Especially to be noticed in this chapter are the formulas of Bonnet and of Enneper relative to the torsion of asymptotic lines, the study of a surface referred to its asymptotic lines, the formulas of Lelieuvre, and the discussion of some particular problems (p. 144). Bianchi defines a moulding surface (*surface moulure, Gesimsflache*) as a surface generated by a plane curve whose plane rolls without slipping on any developable surface. By Darboux and some other writers the term is applied only when the developable surface is cylindrical.

It is not desirable to discuss in detail all the chapters of Bianchi's book, pointing out what the author has omitted that might have been put in, and describing and criticizing his methods. It seems to me that it will give a good idea of the scope and completeness of the book to discuss in considerable detail one chapter of the book, and to compare that with the writings of other authors on the same subject. I have chosen for that purpose the sixth chapter, on geodesic curvature and geodesic lines, because it is both interesting and typical. I shall compare this chapter with the part of Darboux's work which treats the same subject. It is to be remembered that the purposes of the authors are utterly different, and that the standard of comparison is the most difficult that could be set. The comparison is not meant to be in any way disparaging to Bianchi. It is intended merely to show how much about any subject we may expect to find in his book, and to illustrate the choice and coördination of material which he has made.

The sixth chapter of Bianchi's *Vorlesungen* consists of thirty-two pages. Darboux devotes to the same subject about two hundred and forty pages, printed somewhat more closely, besides a discussion of mechanics, based on the treatment of geodesic lines, covering seventy-three pages. Bianchi defines first the geodesic or tangential curvature, and obtains the analytic expression for the tangential curvature of the coördinate lines supposed orthogonal. Following Bonnet, he then obtains by the use of differential parameters the geodesic curvature of any line of the surface, $\varphi(u, v) = 0$. Darboux defines geodesic curvature at first in the same way in a chapter (Livre V., Ch. I.) on "Formules générales," and notes that the center of geodesic curvature is the intersection of the axis of the osculating circle of the curve with the plane tangent to the surface at the point considered. Darboux obtains his expression for the geodesic curvature from kinematical considerations. Later (Livre VI., Ch. VI.) he gives various other definitions and properties of the geodesic curvature, and finally in a chapter on differential parameters (Livre VII., Ch. I.) we find Bonnet's expression for the geodesic curvature of any curve. Bianchi gives next Liouville's expression for the total curvature of a surface at any point expressed in terms of the geodesic curvature of the coördinate lines supposed orthogonal. From this expression may at once be proved the theorem that only on surfaces of constant negative curvature can there exist two orthogonal families of curves having all the same constant geodesic curvature.

Bianchi then defines geodesic lines on a surface as lines whose principal normals coincide always with the normals to the surface. The differential equation of the geodesics is next obtained in several forms. Darboux obtains practically the same equations (Livre V., Ch. IV.), starting again from kinematical considerations. Darboux obtains also the differential equation of the geodesic lines, in a form which is frequently useful, by defining them as the shortest lines of the surface between two points of the surface, and by solving the simple problem in the calculus of variations posed by this definition. I regret that Bianchi does not study at all this problem of the calculus of variations. He merely states its result (p. 154). Darboux next calls attention to the fact that the minimal curves of the surface are geodesic lines in the sense that they satisfy the differential equation of the geodesics. He shows also how they satisfy the first definition given for geodesics. We may note here that Bianchi makes no mention of the minimal curves of a surface, so far as I know, throughout the book. This seems to me unfortunate. For, though these curves are always imaginary, they are of considerable interest, and sometimes of use in solving problems concerning the real elements of surfaces. They have many curious properties which deserve mention in such a book as Bianchi's. Besides being geodesic lines, they are all lines of curvature, and furnish the only exception to the theorem proved by Bianchi (p. 166) that all geodesic lines which are at the same time lines of curvature are plane curves. Another property of these curves is the following, proved, for example, by Raffy, Applications géométriques de l'analyse, p. 147: The only surfaces, not spheres, all of whose points are umbilics are developable surfaces whose edges of regression are minimal curves. All these theorems are omitted by Bianchi.*

A considerable rôle in the theory of geodesic lines is played by the problem of determining in how far the geodesic lines are the shortest lines of the surface between two of their points. Fundamental for this problem is the system of "polar coördinates" consisting of the geodesics radiating from a point of the surface and their orthogonal trajectories. Before such a system can be used as a coördinate system it is necessary to know that any point of a surface can be surrounded by a region in which the geodesics issuing from the point do not again intersect. This is a point assumed by most writers, among them Bianchi. Darboux proves the theorem very carefully (Volume II., p. 408). This theorem assumed, it is easy to prove that the geodesic line is actually the shortest line from the origin of the polar coördinates to any point of the surrounding region. This fact is proved by Bianchi (p. 163). The minimum properties of the geodesics are not further considered by him. Darboux devotes a very interesting chapter of twenty-six pages (Livre VI., Ch. V.) to this subject. This chapter is to a considerable extent the application of Jacobi's work in the calculus of variations to the special problem of the geodesic lines. In §80 Bianchi returns to the differential equation of the geodesics and obtains for it a new form, due to Gauss, in terms of the angle which a geodesic makes with one of the coördinate curves. This equation he uses to give a new definition of geodesic curvature, which is the natural generalization of the curvature of a plane curve. The author next discusses systems of geodesically parallel curves, defining them and obtaining the necessary and sufficient condition that a given family of curves shall be geodesically parallel. In the next section he discusses the subject of geodesic circles. Bianchi defines a geodesic circle as the locus of a point whose geodesic distance from a fixed point is constant; Darboux defines a geodesic circle as a curve of constant geodesic curvature. Both authors call attention to the fact that these definitions are not in general coincident. Neither discusses the question. Bianchi states in a footnote (p. 162), that the two definitions are equivalent for surfaces of constant curvature, but gives no proof. The statement may be proved in a couple of lines, and might perhaps well have been proved. The consideration of this problem leads to the more general theorem that the necessary and sufficient condition that each curve of one family of concentric geodesic circles (in Bianchi's sense) should have constant geodesic curvature is that the total curvature of the surface should be constant along each curve. This may be otherwise stated: The necessary and sufficient condition that there exist on a surface a single family of concentric geodesic circles, such that the geodesic curvature of each circle is constant, is that the surface shall be applicable to a surface of revolution so that the geodesic circles fall on the latitude circles of the surface of revolution.

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* See p. 93.

It is noticeable that Bianchi has very little to say about curves of constant geodesic curvature. In the last section of the chapter (§91) he proves the theorem: A doubly orthogonal system of curves of constant geodesic curvature is always isothermal, and conversely, if one family of curves of an isothermal system be curves of constant geodesic curvature, the curves of the other family will also be of constant geodesic curvature. We have no discussion of the isoperimetrical problem on a surface. The solution of this problem gives, as is known, always a curve of constant geodesic curvature. This and analogous problems are discussed at length by Darboux in a chapter on "Les cercles géodésiques" (Livre VI., Ch. VI.). It seems to me that the subject of curves of constant geodesic curvature is worthy of even fuller discussion than that given by Darboux. It would be interesting, for example, to find what surfaces admit such curves which are in general closed; for on such surfaces the isoperimetrical problem is of especial interest. That these curves are not generally closed appears from the fact that a surface may be passed through any curve without singularities, so that, for that surface, the curve is of constant geodesic curvature. In §83, Bianchi takes up the subject of geodesic conics, and gives a concise treatment of them. He omits the discussion of surfaces on which there exist families of curves which may be regarded in more than one way as geodesic conics. These surfaces are surfaces for which the linear element may be written in the form known by the name of Liouville, as is proved by Darboux (Volume 3, p. 17). Next are taken up the subjects of the torsion of the geodesic line, and of the geodesic torsion of any curve of the surface. In a footnote is added the interesting expression for the torsion of a geodesic line, when the lines of curvature are chosen as coördinate curves. In the next two sections are taken up, briefly, methods of finding the geodesic lines of a surface from an integral of the partial differential equation $\Delta_1\theta = 1$. Though these sections are naturally much less complete than Darboux's treatment of the same subject, we find in them the important results of the theory. These methods are then applied to Liouville's surfaces and to surfaces of revolution. The equation of any geodesic line on a Liouville's surface is $\partial\theta/\partial a = b$, where

$$\theta = \int \sqrt{a(u) + a}\ du \pm \int \sqrt{\beta(v) - a}\ dv,$$

a and b being arbitrary constants. As is proved by Bianchi, the curves (u, v) are at the same time isothermal coördinates and the parameters of families of a system of confocal geodesic conics. It is easily seen that the "ground curves" of the families of conics are any two curves $\theta =$ constant, with opposite connecting signs, where the same value is given to a in both cases. From this it follows, as stated above, that the system of geodesic conics (u, v) may be regarded as conics in an infinite number of ways, *i. e.*, for all values of a. In his discussion of the geodesics on a surface of rotation, Bianchi obtains "Clairaut's equation," $r \sin \psi = k$. In the last paragraph of the section (p. 174) is the obvious slip, for the case where the surface has a maximum circle of latitude: "die Curve verläuft ganz innerhalb der Zone, in der die Radien der Parallelkreise nicht *grosser* sind als k." For *grösser* read *kleiner*. This mistake occurs in the Italian, and was not corrected in the translation. Darboux discusses (Volume 3, pp. 2–9) the determination of surfaces of revolution, all of whose geodesics are closed curves; Bianchi omits this discussion. Finally (§90) we have a very neat and simple proof of Gauss's theorem, that the total curvature of a geodesic triangle is equal to the excess of the sum of its angles over two right angles. Darboux gives a very elaborate discussion of the properties of geodesic triangles based on Gauss's memoir of 1828 (Livre VI., Ch. VIII.).

There are various problems connected with geodesic lines to be found in other parts of the volume. For example, on page 436 is discussed very simply the problem of finding the surfaces which can be represented in such a way on the plane that the geodesics of the surface are represented by the straight lines of the plane. Darboux gives an entirely different proof, longer but simpler in some respects, and discusses the more general problem of the "geodesic representation" of one surface on another.

In closing my account of this chapter I must say that it appears to me a most admirable one. All the principal results of the theory, with almost insignificant exceptions, are given in very small compass and in perfectly clear form.

Of the remaining chapters I can give but a brief summary. In writing this summary I have followed a short review of Bianchi's book published by Darboux in the *Bulletin des Sciences mathématiques*, Volume 32 (1897), p. 253. Chapter VII. contains the fundamental theorems relative to applicable surfaces. Chapter VIII. discusses ruled surfaces, especially the problem of the deformation of such surfaces. In it are given the chief results of Chasles, Bonnet, Minding, and Beltrami. Chapter IX. discusses evolute surfaces, and begins the study of surfaces known as W or Weingarten surfaces, for which there exists a relation between the two principal radii of curvature of the surface at every point. Chapter X. gives the infinitesimal theory of rectilinear congruences. Chapters XI. and XII. treat the infinitesimal deformation of surfaces and the correspondence of surfaces with orthogonal elements. Chapter XIII. discusses "cyclical systems," that is to say congruences of circles normal to a family of surfaces. Chapter XIV. discusses the elements of the theory of minimal surfaces. In Chapter XV. is given a very detailed study of Schwarz's minimal surface bounded by four contiguous edges of a regular tetrahedron. Chapter XVI. studies the conformal representation of a pseudo-spherical surface on the half plane. Chapter XVII. contains a study of the transformations of surfaces of constant curvature. Chapters XVIII., XIX., and XX., the last three chapters of the Italian edition, contain a summary of the theory of orthogonal curvilinear coördinates in space or of triply orthogonal systems of surfaces. There are to be found the theorems of Dupin and Darboux, the theorem of Liouville on the conformal representation of space, the theorem of Combescure on triple systems having a given spherical representation, and the application of the general theory to homofocal surfaces of the second degree, together with a detailed study of the geodesic lines of these surfaces.

Finally is given the investigation, due to Bianchi, of triple systems containing a family of surfaces of constant curvature. In chapters XXI. and XXII., which are published for the first time in the German translation, are given the principal formulas of n-dimensional differential geometry, with especial reference to spaces of constant Riemann's curvature. An appendix to chapter XVII., printed at the end of the *Vorlesungen*, contains a short report of Bianchi's recent researches on the transformation of surfaces of constant positive curvature.

It remains only to say a few words of the book as a whole. It is certainly a magnificent production; the treatment is throughout clear and usually exhaustive. There is not a page that is dull, though there are many that are difficult. If the book has any fault, it is that it is in many places too difficult for the beginner. This fact is recognized by the author who advises the student to omit, in his first reading, certain chapters. A good testimony to the excellence of the book is that Darboux finds in it nothing to criticise but its title.* It is, I think, to be regretted that the author gives so few direct references. A table of authors consulted, printed at the end of the volume, hardly fills their place. A very great convenience is the giving of page references to formulas previously obtained. This very excellent plan is adopted in no other book that I know.

Strange as it may seem, the translation is in many respects an improvement on the original. It is, so far as I can see, an excellent translation. It is printed with better type and on better paper than the original. A heading is printed at the beginning of each section. This makes the page more attractive than that of the original, besides being a considerable convenience. Moreover to the translation has been added a general index. Most of the errors and misprints of the Italian have been corrected, but not all, as I have already noted. A few new errors and misprints have crept in, but none would, I think, inconvenience the reader. It seems not worth while to give a list of these. Certainly Bianchi and his translator, Lukat, have rendered the mathematical public no inconsiderable service in the production of the "Differentialgeometrie."

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* *Bulletin des sciences mathématiques*, vol. 34 (1899), p. 323.

NOTES.

THE third (July) number of Volume II. of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the convergence of continued fractions with complex elements," by E. B. VAN VLECK; "Geometry within a linear spherical complex," by P. F. SMITH; "A new determination of the primitive continuous groups in two variables," by H. F. BLICHFELDT; "Determination of all the groups of order p^m which contain the abelian group of type $(m-2, 1)$, p being any prime," by G. A. MILLER; "Zur linearen Transformation der ϑ-Reihen," by F. MERTENS; "On a fundamental property of a minimum in the calculus of variations, and the proof a theorem of Weierstrass," by W. F. OSGOOD; "Concerning Harnack's theory of improper definite integrals," by E. H. MOORE.

THE July number (volume 23, number 3), of the *American Journal of Mathematics* contains the following articles: "Geometry on the cubic scroll of the second kind," by F. C. FERRY; "Congruent reductions of bilinear forms," by T. J. I'A. BROMWICH; "On the imprimitive substitution groups of degree fifteen, and the primitive substitution groups of degree eighteen," by E. N. MARTIN; "Removal of two terms from a binary quantic by linear transformations," by B. G. MORRISON.

THE July number (second series, volume 2, number 4) of the *Annals of Mathematics* contains the following papers: "Concerning du Bois-Reymond's two relative integrability theorems," by E. H. MOORE; "On a theorem of kinematics," by P. SAUREL; "The collineations of space which transform a non-degenerate quadric surface into itself," by R. G. WOOD; "Note on multiply perfect numbers," by J. WESTLUND; "The isoperimetrical problem on any surface," by J. K. WHITTEMORE; "On a surface of the sixth order which is touched by the axes of all screws reciprocal to three given screws," by E. W. HYDE; "Note sur l'évaluation d'une intégrale définie," by D. SINTSOF.

CORNELL UNIVERSITY.—During the academic year 1901–1902, the following advanced courses will be offered by the department of mathematics, each course extending through the year unless the contrary is explicitly stated: By Professor L. A. WAIT: Advanced analytic geometry, three hours; Advanced differential calculus three hours.—By Professor J. McMahon: Theoretical mechanics, two hours; Potential functions and spherical harmonics, two hours; Theory of sound, two hours.—By Professor J. H. Tanner: Algebraic invariants, two hours; Reading course in German, two hours.—By Professor Trevor: Mathematical theory of thermodynamics, two hours.—By Dr. J. I. Hutchinson: Advanced integral calculus, two hours; Elliptic, automorphic, and abelian functions, three hours.—By Dr. V. Snyder: Projective geometry, three hours; Algebraic curves, three hours; General function theory, three hours. By Dr. W. B. Fite: Introduction to the theory of groups, three hours; Theory of numbers, two hours.—By Mr. ———: Differential equations, three hours.

Johns Hopkins University.—The following advanced courses are offered by the department of mathematics during the academic year 1901–1902:—By Professor F. Morley: Geometry (advanced course), three hours; The differential equations of physics, two hours, first half year; Kinematics, two hours, second half-year; Mathematical seminary, weekly.—By Dr. A. Cohen: Advanced differential equations, two hours; Theory of algebraic numbers, two hours; Elementary theory of functions, two hours; Elementary differential equations, two hours.

Dr. F. Franklin will give a short course on Probability in the second half-year.

Professors David Hilbert, of Göttingen, Georg Cantor, of Halle, and Ulisse Dini, of Pisa, have been elected honorary foreign members of the London mathematical society.

Dr. G. Zeuner, of Dresden, has been elected a corresponding member of the Paris academy of sciences, in the section of mechanics.

Dr. A. S. Chessin, formerly of Johns Hopkins University, has been appointed professor of mathematics at Washington University, St. Louis, Mo., to succeed Professor E. A. Engler, who was recently elected president of the Worcester Polytechnic Institute.

At Harvard University Mr. J. L. Love has been promoted to an assistant professorship of mathematics.

Dr. W. B. Fite has been promoted to an instructorship in mathematics at Cornell University.

The death is announced of Dr. P. Helmling, formerly professor of mathematics at the University of Dorpat, at the age of eighty-four.

Professor C. S. James, formerly professor of mathematics and physics in Bucknell College, died June 8, 1901.

Professor T. H. Safford, of Williams College, died June 13, 1901, aged sixty-seven years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Adam (C.). See Descartes (R.).

Amaldi (U.). See Pincherle (S.).

Andreief (K. A.). Elements of analytic geometry, with exercises and solutions. Moscow, 1900. 8vo. 216 pp. (Russian.) M. 10.00

Arbicone (A.). See Forsyth (A. R.).

Autenheimer (F.). Elementarbuch der Differential- und Integralrechnung, mit zahlreichen Anwendungen aus der Analysis, Geometrie, Mechanik und Physik. Für höhere Lehranstalten und den Selbstunterricht. 5te Auflage, bearbeitet von A. Donadt. Leipzig, Voigt, 1901. 8vo. 10 + 602 pp. M. 9.00

Borel (E.). Leçons sur les séries divergentes. Paris, Gauthier-Villars, 1901. 8vo 6 + 186 pp. (Nouvelles leçons sur la théorie des fonctions, III.) Fr. 4.50

Cantor (M.). Vorlesungen uber Geschichte der Mathematik. (In 3 Bänden.) Vol. III, Abteilung 2: von 1700–1726. 2te Auflage. Leipzig, Teubner, 1901. 8vo. 2 + 218 pp. M. 3.20

Descartes (R.). Oeuvres, publiées par C. Adam et P. Tannery, sous les auspices du ministère de l'instruction publique. Correspondance. Vol. IV : juillet 1643–april 1647. Paris, Cerf, 1901. 4to. 713 pp.

Donadt (A.). See Autenheimer (F.).

Edser (E.). Differential and integral calculus; for beginners, adapted to the use of students of physics and mechanics. London, Nelson, 1901. 8vo. 254 pp. 2s. 6d.

Ellensson (L.). Die reciproken Polarkurven der Epi- und Hypocykloiden. (Diss.) Bern, 1899. 8vo. 103 pp., 1 plate.

Forsyth (A. R.). Trattato sulle equazioni differenziali. Prima versione dall' inglese da A. Arbicone. Livorno, 1901. 8vo. 352 pp. Fr. 8.50

Geck (E.). Ueber die singulären Punkte algebraischer Flächen. (Diss). Tubingen, 1900. 8vo. 40 pp.

Hamilton (W. R.). Elements of quaternions. 2d edition by C. J. Joly. Vol. II. London, Longmans, 1901. 8vo. 21s.

JOLY (C. J.). See HAMILTON (W. R.).

KIEPERT (L.). Tabelle der wichtigsten Formeln aus der Differentialrechnung. Hannover, Helwing, 1901. 8vo. 34 pp. M. 0 50

KROHS (G.). Die algebraisch lösbaren irreduciblen Gleichungen funften Grades. Teil I. Berlin, 1900. 4to. M. 1.00

LOREY (W.). Ueber das geometrische Mittel, insbesondere uber eine dadurch bewirkte Annäherung kubischer Irrationalitaten. (Diss.) Remscheid, Krumm, 1901. 4to. 27 pp. M. 1.00

OPITZ (H. R. G.). Studie uber die Rudio'schen Flachen. Berlin, 1901. 4to. M. 1.00

PALÁGYI (M.). Neue Theorie des Raumes und der Zeit; die Grundbegriffe einer Metageometrie. Leipzig, 1901. 8vo. M. 0.80

PICARD (E.). Traité d'analyse. 2e édition, revue et corrigée. Vol. I: Intégrales simples et multiples; l'équation de Laplace et ses applications; développements en séries; applications géométriques du calcul infinitésimal. Paris, Gauthier-Villars, 1901. 8vo. 15 + 484 pp. Fr. 16 00

PINCHERLE (S.) e AMALDI (U.). Le operazioni distributive e le loro applicazioni all'analisi. Bologna, Zanichelli, 1901. 8vo. 12 + 490 pp. Fr. 15.00

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RUNFER (A.). Ueber Schraubenlinien und Schraubenflachen. (Diss.) Bern, 1900. 4to. 68 pp., 8 plates.

ZIEGEL (R.). Die lineare Differentialgleichung dritter Ordnung mit algebraischen Integralen. (Diss.) Berlin, 1900. 8vo. 25 pp.

II. ELEMENTARY MATHEMATICS.

AGAPOF (D. V.). Solution and discussion of the more difficult arithmetical problems in the collections of A. Malinin, K. Burenin, and Vereshchagin. Orenburg, 1900. 8vo. 273 pp. (Russian). M. 3.50

BACKHAUS (K.). See WIESE (B.).

BENDT (F.). Katechismus der Trigonometrie. Leipzig, 1901. 8vo. 8 + 135 pp. M. 2 00

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BOYMAN (I. R.) See FÜCHTJOHANN (H.).

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FÜCHTJOHANN (H.). Losung der Aufgaben in I. R. Boyman's Lehrbuch der Planimetrie. Teil I. Aufgabe 1–733. Bonn, Cohen, 1901. 8vo. 3 + 160 pp. M. 3 60

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GAMBIOLI (D.). Calcolo litterale ed equazioni di primo grado, con appendice sulle radici quadrata e cubica. 2a edizione, ampliata e migliorata. Bologna, 1901. 12mo. 248 pp. Fr. 2 50

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GREMIGNI (M.). Elementi di geometria ad uso delle scuole tecniche e professionali. Planimetria, libro I–II. 2a edizione. Firenze, Bemporad, 1901. 16mo. 11 + 154 pp. (Biblioteca scolastica.) Fr. 1 20

HANDEL (O.). Elementar-synthetische Kegelschnittslehre. Mit zahlreichen Uebungsaufgaben. Zum Gebrauch an hoheren Lehranstalten bearbeitet. 2te Auflage. Berlin, Weidmann, 1901. 8vo. 6 + 85 pp. Board. M. 1.60

HÄNTZSCHEL (E.). Elementare Herleitung der Newton'schen Reihen für Sinus und Cosinus. Berlin, 1901. 4to. M. 1.00

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